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GRAPHOID PROPERTIES OF QUALITATIVE POSSIBILISTIC INDEPENDENCE RELATIONS

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Independence relations play an important role in uncertain reasoning based on Bayesian networks. In particular, they are useful in decomposing joint distributions into more elementary local ones. Recently, in a possibility theory framework, several qualitative independence relations have been proposed, where uncertainty is encoded by means of a complete pre-order between states of the world. This paper studies the well-known graphoid properties of these qualitative independences. Contrary to the probabilistic independence, several qualitative independence relations are not necessarily symmetric. Therefore, we also analyze the symmetric counterparts of graphoid properties (called reverse graphoid properties)

Keywords: Possibilistic independence; plausibility relations; graphoid properties.

1. Introduction

Independence relations play an important role in handling uncertain information. Two forms of independences can be distinguished: *causal* relations which express the lack of causality between variables and *decompositional* ones which ensure the decomposition of a joint distribution pertaining to tuples of variables into local distributions on smaller subsets of variables. Causal independences are not necessarily symmetric contrary to decompositional ones.

In the probabilistic framework, two variables A and B are said to be decomposably independent if the joint probability distribution on the range of (A, B) is the product of the probability distribution of A and the probability distribution of B, i.e., $P(A \wedge B) = P(A) \cdot P(B)$. Moreover, A and B are said to be causally independent if the probability of B given A is the same as the probability of B, i.e., $P(B \mid A) = P(B)$. In this framework causal and decompositional independence relations are equivalent.

In possibility theory, and more generally in total pre-orderings settings, the situation is different since causal and decompositional relations are not always equivalent. In[2] several forms of qualitative independence relations have been proposed in possibility theory framework. These new relations are only based on the qualitative plausibility relations induced by possibility distributions.

This paper goes one step further by studying graphoid properties[7,22] of qualitative independence relations proposed in[2]. Graphoid properties are very important in the study of Bayesian networks. In particular, they are useful in developing efficient local propagation algorithms.

The graphoid properties are dedicated to symmetric independence relations. For instance, the decomposition property asserts that if X is independent of $Y \cup W$ (by symmetry $Y \cup W$ is independent of X), then X is independent of Y (resp. W) and by symmetry Y (resp. W) is also independent of X.

However, if an independence relation is not symmetric, then the decomposition property simply states that if $Y \cup W$ is irrelevant to X, then Y (resp. W) is irrelevant to X too. Namely, a non-symmetric relation which satisfies the decomposition property *may not* allow to conclude that X is irrelevant to Y (resp. W) from the fact that $Y \cup W$ is irrelevant to X.

This paper also proposes to analyze qualitative possibilistic independence relations with respect to symmetric counterparts of graphoid properties called *reverse graphoid properties*[25].

Section 2 gives a brief background on possibility theory. Section 3 recalls recent causal and decompositional qualitative independence relations proposed in[2]. Section 4 recalls graphoid properties. Section 5 and Section 6 study graphoid properties of non-symmetric and symmetric independence relations, respectively. Lastly, Section 7 summarizes main results regarding graphoid properties. Proofs are provided in the Appendix.

2. Basics of possibility theory

2.1. *Notations*

Let $V = \{A_1, A_2, ..., A_N\}$ be a set of variables. We denote by D_A the supposedly finite domain associated with the variable A. By a_i we denote any instance of A_i. $X, Y, Z, ...$ denote subsets of variables from V, and $D_X = \times_{A_i \in X} D_{A_i}$ represents the Cartesian product of domains of variables in X. D_A (resp. D_X) is also called the range of the variable A (resp. the set of variables X). By x we denote any instance of X; if $X = \{A_1, ..., A_n\}$ then $x = (a_1, ..., a_n)$ denotes an instance of D_X. $\Omega = \times_{A_i \in V} D_{A_i}$ denotes the universe of discourse, which is the Cartesian product of all variable domains in V. Each element $\omega \in \Omega$ is called a possible world, elementary event or state of Ω. Depending on the context, we use one of the following notations: either tuples: $\omega = (a_1, ..., a_N)$ or conjunctions: $\omega = a_1 \wedge ... \wedge a_N$.
ϕ, ψ denote subsets of Ω (called propositions or events) and $\neg\phi$ denotes the complementary set of ϕ, namely, $\neg\phi = \Omega - \phi$. $\phi \wedge \psi$ denotes the intersection of ϕ and

ψ.

2.2. *Possibility and necessity measures*

This subsection gives a brief recalling on possibility theory, for more details see[15].

A first notion in possibility theory is the one of *possibility distribution*. It is a mapping from Ω to the scale [0, 1] usually denoted by π. Possibility distributions aim at encoding an agent's knowledge about an ill-known world : $\pi(\omega) = 1$ means that ω is totally possible and $\pi(\omega) = 0$ means that ω can not be the real world. A possibility distribution π is said to be *normalized* if there exist at least one state ω which is totally possible.

Given a possibility distribution π, the uncertainty of any event $\phi \subseteq \Omega$ is characterized by means of two dual measures:

- The **possibility measure** of ϕ (which is a basic notion in a possibility theory):
$$\Pi(\phi) = \max_{\omega \in \phi} \pi(\omega). \quad (1)$$
The measure $\Pi(\phi)$ evaluates at which level ϕ is **consistent** with our knowledge represented by the possibility distribution π.
- The **necessity measure**, associated with Π by duality:
$$N(\phi) = 1 - \Pi(\neg\phi) = \min_{\omega \notin \phi}(1 - \pi(\omega)). \quad (2)$$
The measure $N(\phi)$ corresponds to the extent to which $\neg\phi$ is impossible and thus evaluates at which level ϕ is **certainly** implied by our knowledge (represented by the possibility distribution π).

2.3. *Possibilistic conditioning*

Conditioning is a crucial notion when studying independence relations. In the possibilistic setting it consists in modifying our initial knowledge on X encoded by the possibility distribution π by the arrival of the event $[Y = y]$. The initial distribution π is then replaced by another one denoted by $\pi' = \pi(. \mid y)$.

In possibility theory there are several definitions of conditioning[3,4,10,14,19,21] (see also[26] for an overview of existing possibilistic independence relation).

In this section, in order to easily define independence relation, conditioning is given in terms of possibility measures, instead of possibility distributions.

Possibilistic conditioning $\Pi(x \mid y)$ is generally derived from $\Pi(x \wedge y)$ and $\Pi(y)$, following an equation close to the Bayesian rule, of the form:

$$\forall x, \Pi(x \wedge y) = \Pi(x \mid y) \otimes \Pi(y). \quad (3)$$

where $\otimes$ is a t-norm.

When using the *minimum* (or Gödel's) t-norm and the *product* t-norm as examples of $\otimes$ in 3, we get:

- *min-based conditioning* proposed by Hisdal[21] (see also[15]).

However, as noticed by de Cooman [10], the definition of conditional possibility distribution is not uniquely defined. The solution proposed by Dubois and Prade[15] consists in considering the following greatest solution (least specific conditional possibility distribution) to:

$$\Pi(x \mid_m y) = \begin{cases} 1 & if\ \Pi(x \wedge y) = \Pi(y) \\ \Pi(x \wedge y) & if\ \Pi(x \wedge y) < \Pi(y) \\ 0 & otherwise. \end{cases} \tag{4}$$

If $\Pi(y) = 0$ then, by convention $\Pi(x \mid_m y) = \Pi(x \mid_p y) = 1$.

- *product-based conditioning* proposed in a numerical setting and which is a direct counterpart of probabilistic conditioning (for $\Pi(y) \neq 0$):

$$\Pi(x \mid_p y) = \frac{\Pi(x \wedge y)}{\Pi(y)}. \tag{5}$$

Note that the product-based conditioning (4)) is equivalent to the Dempster rule of conditioning [12].

Fonck[18] also gives another definition of conditioning based on the Lukasziewicz' t-norm. The conditioning rule is then:

$$\Pi(x \mid_l y) = \Pi(x \wedge y) - \Pi(y) + 1. \tag{6}$$

However, the main limitation of this definition is that an impossible event can become somewhat possible after conditioning.

There exist other definitions of conditioning. For instance de Campos et al. [8,9] proposed the following definition which is a modification of the min-based conditioning (4):

$$\Pi(x \mid_{hc} y) = \begin{cases} \Pi(x) & \text{if } \Pi(x \mid_m y) \geq \Pi(x) \forall x \in D_X \\ \Pi(x \mid_m y) & \text{if } \exists x' \in D_X \text{ s.t } \Pi(x' \mid_m y) < \Pi(x') \end{cases} \tag{7}$$

In addition, they proposed a modification of the product-based conditioning (5) which is more restrictive:

$$\Pi(x \mid_{dc} y) = \begin{cases} \Pi(x) & \text{if } \Pi(x \wedge y) \geq \Pi(x) \cdot \Pi(y) \forall x \in D_X \\ \Pi(x \mid_p y) & \text{if } \exists x' \in D_X \text{ s.t } \Pi(x' \wedge y) < \Pi(x') \cdot \Pi(y) \end{cases} \tag{8}$$

The idea behind these two definitions is that if after conditioning we obtain a less informative distribution, then it is better to use the unconditional distribution in order to not loose any information.

Lastly, and contrary to most existing works where conditioning $\Pi(x \mid y)$ is defined from $\Pi(x \wedge y)$ and $\Pi(y)$, Bouchon-Meunier et al.[4] consider conditional possibility as a primitive concept which is directly defined as a function whose domain is a set of conditional events $x \mid y$, with $y \neq \emptyset$.

More precisely, given a set $\mathcal{C} = \mathcal{X} \times \mathcal{Y}$ of conditional events $x_i \mid y_j$, such that $\mathcal{C}$ is a Boolean algebra, $\mathcal{Y}$ an additive set, with $\mathcal{Y} \subseteq \mathcal{X}$, and $\emptyset \notin \mathcal{Y}$, then a function Π on $\mathcal{C}$ is a $\otimes$-*conditional possibility* if the following conditions hold:

(i) $\Pi(X \mid Y) = \Pi(X \wedge Y \mid Y), \forall X \in \mathcal{X}$ and $Y \in \mathcal{Y}$;

(i) $\Pi(. \mid Y)$ is possibility measure, for any given $Y \in \mathcal{Y}$;
(iii) $\Pi(X \wedge A \mid Y) = \Pi(X \mid Y) \otimes \Pi(A \mid X \wedge Y)$, $\forall A \in \mathcal{X}$, $Y \in \mathcal{Y}$, $X \wedge Y \in \mathcal{Y}$ for a triangular norm $\otimes$.

2.4. *Existing possibilistic independence relations*

As we have seen in the previous subsection, there exist multiple definitions of conditioning in the possibilistic framework. This leads to several definitions of possibilistic independence.
Different works have been achieved on this topic[4,8,9,11,17,18,19,24,26,27]. This subsection gives a brief refresher on existing possibilistic independence relations.

In the rest of this paper, given three mutually disjoint subsets of variables X, Y and Z of V, we use the notation $I(X, Y \mid Z)$ to say that X is independent of Y in the context of Z.

One natural way to define independence relations in the possibilistic setting is to consider that X is independent from Y in the context Z, if for any instance $z \in D_Z$, the possibility degree of any $x \in D_X$ remains unchanged for any value $y \in D_Y$. Namely, $\forall x \in D_X, y \in D_Y, z \in D_Z$:

$$\Pi(x \mid y \wedge z) = \Pi(x \mid z). \tag{9}$$

Since possibility theory admits several definitions of conditioning, this leads to several definitions of causal possibilistic independence obtained by replacing the conditioning in (9) by different forms of conditioning (i.e. (4) (5) (6) (7) (8)).

For sake of simplicity, we only develop the min-based and product-based independence relations:

- **Min-based independence relation** obtained by using the min-based conditioning (4) in (9). This form of independence, denoted by I_M, is not symmetric i.e. $I_M(X, Y \mid Z) \neq I_M(Y, X \mid Z)$ where Z denotes the context variable, as pointed out by Fonck[18].
  Let us denote $I_{MS}(X, Y \mid Z)$ the symmetrized version of I_M suggested in [17] (called MS-independence), defined by $I_{MS}(X, Y \mid Z)$ iff
  $\forall x \in D_X, \forall y \in D_Y, \forall z \in D_Z$:

  $$\textbf{(i) } \Pi(x \mid_m y \wedge z) = \Pi(x \mid_m z) \text{ and}$$
  $$\textbf{(ii) } \Pi(y \mid_m x \wedge z) = \Pi(y \mid_m z). \tag{10}$$

  This relation is a very restrictive one since the MS-independence between two sets of variables X and Y requires full ignorance about one of them (uniform distribution) [8,9] i.e. $\Pi(x) = 1, \forall x \in D_X$ or $\Pi(y) = 1, \forall y \in D_Y$.
- **Product independence relation** obtained by using the product-based conditioning (5) in (9). This form of independence, denoted I_P, can be written using $\forall x \in D_X, y \in D_Y, z \in D_Z$:

  $$\Pi(x \wedge y \mid_p z) = \Pi(x \mid_p z) \cdot \Pi(y \mid_p z). \tag{11}$$

Similarly, when conditional possibility is directly defined on conditional events, Bouchon-Meunier et al.[4] define conditional independence of X and Y in the context of Z, denoted by $I_{CE}(X, Y \mid Z)$ (CE for Conditional Events), iff for any events $x \in D_X, y \in D_Y, z \in D_Z$ we have:

$$\Pi(x \mid y \wedge z) = \Pi(x \mid z). \tag{12}$$

This independence relation is not symmetric, and Bouchon-Meunier et al.[4] have defined its symmetric counterpart, denote $I_{SCE}(X, Y \mid Z)$, as simply:

$$I_{SCE}(X, Y \mid Z) \text{ iff } I_{CE}(X, Y \mid Z) \text{ and } I_{CE}(Y, X \mid Z). \tag{13}$$

de Campos et al.[8,9] propose an independence definition of X and Y in the context of Z, if given any value of Z, if we know the value that Y takes, we obtain a piece of information about X *similar* to the one prior learning the value of Y. More formally $\forall x \in D_X, \forall y \in D_Y, \forall z \in D_Z$:

$$\Pi(x \mid y \wedge z) \approx \Pi(x \mid z). \tag{14}$$

This definition was studied in [8,9] by using the min-based conditioning (4) and the product-based conditioning (5).

Alternative definitions of possibilistic independence were suggested by the same authors in[8,9] by replacing the equality in (9) by less restrictive operators. In fact, the independence of X from Y in the context Z is asserted when we do *not gain additional information* about the values of X after conditioning to Y. More formally $\forall x \in D_X, y \in D_Y, z \in D_Z$:

$$\Pi(x \mid y \wedge z) \geq \Pi(x \mid z). \tag{15}$$

This definition, in fact, is equivalent to the standard decompositional independence between X and Y in the context Z is represented by the **non-interactivity** relation introduced by Zadeh[27] for unconditional independence and extended by Fonck for conditional independence, denoted by $I_{NI}(X, Y \mid Z)$ (NI for Non Interactivity) and defined by:

$$\Pi(x \wedge y \mid_m z) = min(\Pi(x \mid_m z), \Pi(y \mid_m z)), \forall x, y, z, \tag{16}$$

or equivalently by[17]:

$$\Pi(x \wedge y \wedge z) = min(\Pi(x \wedge z), \Pi(y \wedge z)), \forall x, y, z. \tag{17}$$

3. Qualitative possibilistic independence

This section recalls recent qualitative independence relations introduced in[2] where two forms of independence (causal and decompositional) have been proposed (for more details see[2]).

The main difference between qualitative possibilistic independence relations and existing ones (recalled in Section 2) is that qualitative possibilistic independence only use plausibility relations induced by possibility distributions. Hence, the interval $[0, 1]$ is used in this section as a mere ordinal scale.

We first need to give a formal description of the qualitative representation of uncertainty we are using, and introduce the concept of accepted beliefs.

3.1. *Basics definitions of qualitative possibility theory*

The basic idea of qualitative possibility distributions is to equip the referential Ω with a *complete pre-order* instead of using the interval $[0, 1]$. This complete pre-order denoted $\geq_\pi$, corresponds to a *plausibility relation* on Ω and simply enables us to express that some situations are more plausible than others.

We denote $=_\pi$ (resp. $>_\pi$, $<_\pi$) the equality (resp. inequality) relation corresponding to $\geq_\pi$. Namely the relation $\omega =_\pi \omega'$ means that ω is as plausible as ω'.

Let $\varphi = \{\omega_1, .., \omega_n\} \subseteq \Omega$ be a subset of Ω, the *most plausible state(s)* (called also normal states) in φ, denoted by $max_{\geq_\pi}(\varphi)$ and is defined as:

$$max_{\geq_\pi}(\varphi) = \{\omega_i : \omega_i \in \varphi, \nexists \omega_j \in \varphi \text{ s.t. } \omega_j >_\pi \omega_i\}. \tag{18}$$

Given a plausibility relation $\geq_\pi$ on Ω, we can lift it to another plausibility relation defined on the subsets of Ω denoted $\geq_\Pi$ by (e.g.,[13]):

$$\phi \geq_\Pi \psi \text{ iff } \forall \omega \in \psi, \exists \omega' \in \phi \text{ such that } \omega' \geq_\pi \omega. \tag{19}$$

Namely, $\phi \geq_\Pi \psi$ holds if there exist a state within the *most plausible state(s)* in ϕ which is preferred to any element in the *most plausible state(s)* in ψ. In other terms:

$$\phi \geq_\Pi \psi \text{ iff } \exists \omega \in max_{\geq_\pi}(\phi) \text{ such that } \forall \omega' \in max_{\geq_\pi}(\psi), \omega \geq_\pi \omega'.$$

The idea behind the relation $\geq_\Pi$ is that the agent whose epistemic state is modeled by the plausibility relation $\geq_\pi$ evaluates events by their most plausible state considering that if ϕ occurs, then the expected situation is among the states in $max_{\geq_\pi}(\phi)$, because they are considered as normal states.

Qualitative conditioning: In the qualitative setting, conditioning consists in focusing a plausibility relation $\geq_\pi$ on a subclass $\phi \subseteq \Omega$, on the basis of a new piece of sure information about a case at hand. A plausibility relation restricted to ϕ, denoted by $\geq_{\pi|\phi}$ is uniquely defined using the following postulates:

$\mathbf{A}_1$**:** $\forall \omega_1, \omega_2 \in \phi, \omega_1 >_\pi \omega_2$ iff $\omega_1 >_{\pi|\phi} \omega_2$,
$\mathbf{A}_2$**:** $\forall \omega_1 \in \phi, \forall \omega_2 \notin \phi, \omega_1 >_{\pi|\phi} \omega_2$,
$\mathbf{A}_3$**:** $\forall \omega_1, \omega_2 \notin \phi, \omega_1 =_{\pi|\phi} \omega_2$.

$\mathbf{A}_1$ means that the new plausibility relation should not alter the initial order between elements of ϕ. $\mathbf{A}_2$ confirms that each element of ϕ should be preferred to any element not belonging to ϕ. Finally, the last postulate $\mathbf{A}_3$ says that elements not belonging to ϕ are irrelevant and should be in the same equivalence class.

We denote $=_{\pi|\phi}$ (resp. $>_{\pi|\phi}$, $<_{\pi|\phi}$) the equality (resp. inequality) relation corresponding to $\geq_{\pi|\phi}$.

The notion of qualitative conditioning extends the possibilistic conditioning recalled in Section 2.2. Indeed, when using possibilistic conditioning on a positive possibility distribution π (with the minimum operator or the product operator) the order of instances in the new conditional possibility distribution is the same as in the conditional plausibility relation computed from the plausibility relation induced from π^1.

Accepted beliefs : We now introduce the notion of accepted beliefs, already used in the context of default reasoning [16,20], and which will be helpful in defining qualitative independence. The acceptance function associated with a plausibility relation $\geq_\pi$ denoted by $\mathbf{Acc}_{\geq_\pi}(.)$ assigns to each ϕ a value in $\{-1, 0, 1\}$ in the following way:

$$\mathbf{Acc}_{\geq_\pi}(\phi) = \begin{cases} 1 & if\ \phi >_\Pi \neg\phi \\ -1 & if\ \neg\phi >_\Pi \phi \\ 0 & if\ \phi =_\Pi \neg\phi. \end{cases} \tag{20}$$

When $\mathbf{Acc}_{\geq_\pi}(\phi) = 1$ (resp. $\mathbf{Acc}_{\geq_\pi}(\phi) = -1$) we say that ϕ is *accepted* (resp. *rejected*). $\mathbf{Acc}_{\geq_\pi}(\phi) = \mathbf{Acc}_{\geq_\pi}(\neg\phi) = 0$, corresponds to the situation of total ignorance concerning ϕ, i.e., ϕ and $\neg\phi$ are equally plausible.

The function $\mathbf{Acc}_{\geq_\pi}$ can be extended in order to take into account a given context. Then a conditional belief measure denoted by $\mathbf{Acc}_{\geq_\pi}(.|.)$ is defined by:

$$\mathbf{Acc}_{\geq_\pi}(\phi \mid \psi) = \begin{cases} 1 & if\ \phi \wedge \psi >_\Pi \neg\phi \wedge \psi \\ 0 & if\ \phi \wedge \psi =_\Pi \neg\phi \wedge \psi \\ -1 & if\ \neg\phi \wedge \psi >_\Pi \phi \wedge \psi. \end{cases} \tag{21}$$

In the following, we use $\mathbf{Acc}(.)$ (resp. $\mathbf{Acc}(.|.)$) instead of $\mathbf{Acc}_{\geq_\pi}(.)$ (resp. $\mathbf{Acc}_{\geq_\pi}(.|.)$) when there is no ambiguity.

3.2. *Causal qualitative independence*

The causal qualitative independence can be seen from different points of view. Namely, the variable set X is independent of Y if upon learning any instance of Y:
- the agent's beliefs on D_X, i.e. the accepted (resp. rejected and ignored) instances of X, are preserved or
- the relative ordering between instances of X is preserved.

In the following, we reproduce the same notations and independence relations names as the ones used in[2].

- Belief-preserving independence: The first notion of causal independence is concerned with the preservation of accepted and rejected beliefs. A set of variables X can be considered as independent of Y in the context Z, if the accepted and rejected beliefs pertaining to X, held in the context Z, remain unchanged when some information about Y is obtained. Formally:

Definition 1. (**BP-independence**) Let $\geq_\pi$ be a plausibility relation defined on Ω and consider three mutually disjoint subsets of variables X, Y and Z of V. The variable set X is said to be BP-independent (BP for Belief Preserving) of Y in the context Z, denoted $I_{BP}(X, Y \mid Z)$, iff $\forall x \in D_X, \forall y \in D_Y, \forall z \in D_Z$:

$$\mathbf{Acc}(x \mid y \wedge z) = \mathbf{Acc}(x \mid z). \tag{22}$$

The BP-independence relation is not symmetric as it will be shown later (Section 5.2). We denote by I_{BPS} the symmetrized version[1] of BP-independence relation; i.e. the variable set X is said to be BPS-independent of Y in the context Z if $\forall x \in D_X, \forall y \in D_Y, \forall z \in D_Z$:

$$\begin{aligned}&\textbf{(i) } \mathbf{Acc}(x \mid y \wedge z) = \mathbf{Acc}(x \mid z) \text{ and}\\ &\textbf{(ii) } \mathbf{Acc}(y \mid x \wedge z) = \mathbf{Acc}(y \mid z).\end{aligned} \tag{23}$$

- Preserving-ordering independence: The second causality-oriented definition says that X is independent of Y in the context of Z, if for all $z \in D_Z$, the local *preferential ordering* between the different instances of X is preserved after the revision by any instance y of Y. More formally:

Definition 2. (**PO-independence**) Let $\geq_\pi$ be a plausibility relation defined on Ω and consider three mutually disjoint subsets of variables X, Y and Z of V. The variable set X is said to be PO-independent (PO for Preserving Ordering) of Y in the context Z, denoted $I_{PO}(X, Y \mid Z)$, if $\forall y \in D_Y, \forall z \in D_Z$:

$$\forall x_i, x_j \in D_X, x_i \wedge z >_\Pi x_j \wedge z \text{ iff } x_i \wedge y \wedge z >_\Pi x_j \wedge y \wedge z. \tag{24}$$

This relation is not symmetric as it will be shown later. We denote I_{POS} the symmetrized version of I_{PO}; i.e. X is said to be POS-independent of Y in the context Z if $\forall x \in D_X, \forall y \in D_Y, \forall z \in D_Z$:

$$\begin{aligned}&\textbf{(i) } \forall x_i, x_j \in D_X, x_i \wedge z >_\Pi x_j \wedge z \text{ iff } x_i \wedge y \wedge z >_\Pi x_j \wedge y \wedge z, \text{ and}\\ &\textbf{(ii) } \forall y_k, y_l \in D_Y, y_k \wedge z >_\Pi y_l \wedge z \text{ iff } x \wedge y_k \wedge z >_\Pi x \wedge y_l \wedge z.\end{aligned} \tag{25}$$

3.3. *Decompositional independence*

This section proposes two classes of decompositional independences, the first is based on belief decomposition and the second on remarkable plausibility relations.

- Belief decompositional independence: The idea of this independence relation is to consider two variable sets X and Y as independent in the context Z if for any instance z of Z, the acceptance of any instance $(x \wedge y)$ of $X \cup Y$ is fully determined by the acceptance of x and y.

Definition 3. (**PT-independence**) Let $\geq_\pi$ be a plausibility relation defined on Ω and consider three mutually disjoint subsets of variables X, Y and Z of V. The variable set X is said to be PT-independent (PT for Preserving Top elements) of Y in the context Z, denoted $I_{PT}(X, Y \mid Z)$, iff $\forall x \in D_X, \forall y \in D_Y, \forall z \in D_Z$:

$$\mathbf{Acc}(x \wedge y \mid z) = min(\mathbf{Acc}(x \mid z), \mathbf{Acc}(y \mid z)). \tag{26}$$

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[1]In what follows the suffix S is used to denote the symmetrized version of non symmetric relations.

- Decompositional independence of remarkable plausibility relations: A plausibility relation is said to be decomposable w.r.t. X and Y in the context Z, iff $\geq_\pi$ is a function of the local orderings on $X \cup Z$ and $Y \cup Z$. The following introduces well known example of orderings used in the qualitative setting.

(i) A plausibility relation $\geq_\pi$ is said to be **Pareto-decomposable** on X and Y in the context Z, if $\forall z \in D_Z, \forall x_i, x_j \in D_X, \forall y_k, y_l \in D_Y$, we have:
$x_i \wedge y_k \wedge z \geq_\pi x_j \wedge y_l \wedge z$ **if and only if** $x_i \wedge z \geq_\Pi x_j \wedge z$ and $y_k \wedge z \geq_\Pi y_l \wedge z$.

(ii) A plausibility relation $\geq_\pi$ is said to be **leximin-decomposable** on X and Y in the context Z, if $\forall z \in D_Z, \forall x_i, x_j \in D_X, \forall y_k, y_l \in D_Y$, we have:
- $x_i \wedge y_k \wedge z >_\Pi x_j \wedge y_l \wedge z$ **if and only if**
(i) $min(x_i \wedge z, y_k \wedge z) >_\Pi min(x_j \wedge z, y_l \wedge z)$ or
(ii) $min(x_i \wedge z, y_k \wedge z) =_\Pi min(x_j \wedge z, y_l \wedge z)$ and
$max(x_i \wedge z, y_k \wedge z) >_\Pi max(x_j \wedge z, y_l \wedge z)$.
- $x_i \wedge y_k \wedge z =_\Pi x_j \wedge y_l \wedge z$ **if and only if**
$min(x_i \wedge z, y_k \wedge z) =_\Pi min(x_j \wedge z, y_l \wedge z)$ and $max(x_i \wedge z, y_k \wedge z) =_\Pi max(x_j \wedge z, y_l \wedge z)$.

where
$$max(a, b) = \begin{cases} a \text{ if } a \geq_\Pi b \\ b \text{ otherwise} \end{cases}$$
and
$$min(a, b) = \begin{cases} a \text{ if } a \leq_\Pi b \\ b \text{ otherwise} \end{cases}.$$

(iii) A plausibility relation $\geq_\pi$ is said to be **leximax-decomposable** on X and Y in the context Z, if $\forall z \in D_Z, \forall x_i, x_j \in D_X, \forall y_k, y_l \in D_Y$, we have:
- $x_i \wedge y_k \wedge z >_\Pi x_j \wedge y_l \wedge z$ **if and only if**
(i) $max(x_i \wedge z, y_k \wedge z) >_\Pi max(x_j \wedge z, y_l \wedge z)$ or
(ii) $max(x_i \wedge z, y_k \wedge z) =_\Pi max(x_j \wedge z, y_l \wedge z)$ and
$min(x_i \wedge z, y_k \wedge z) >_\Pi min(x_j \wedge z, y_l \wedge z)$.
- $x_i \wedge y_k \wedge z =_\Pi x_j \wedge y_l \wedge z$ **if and only if**
$min(x_i \wedge z, y_k \wedge z) =_\Pi min(x_j \wedge z, y_l \wedge z)$ and $max(x_i \wedge z, y_k \wedge z) =_\Pi max(x_j \wedge z, y_l \wedge z)$.

Definition 4. (**Pareto, leximin, leximax-independences**) X and Y are said to be **Pareto-independent** (resp. **leximin-independent**, **leximax-independent**) in the context Z, denoted I_{Pareto} (resp. $I_{leximin}$, $I_{leximax}$), if the plausibility relation $\geq_\pi$ is Pareto-decomposable (resp. leximin-decomposable, leximax-decomposable) on X and Y in the context Z.

3.4. *Summary*

Figure 1 (a) (resp. (b)) illustrates the existing links between the different symmetric (resp. non-symmetric) independence relations (see[2] for proofs). The arrows show the inclusion between independence relations (transitivity is not explicit for sake of clarity). The absence of arrows implies the incomparability of the independence relations. I_{MS} and I_{Pareto} are the strongest independence relations while I_{PT} is the weakest one.

Fig. 1. Links between symmetric (a) and non-symmetric (b) independence relations

4. Graphoid properties

Independence relations can be characterized by the well known graphoid properties which have been largely studied in the probabilistic framework[5,7,22,23]. These properties are as follows:

- *P1: Symmetry* : $I(X, Y \mid Z) \Rightarrow I(Y, X \mid Z)$
  This relation asserts that in any state of context Z, if Y tells us nothing new about Y, then X tells us nothing new about Y.
- *P2: Decomposition*: $I(X, Y \cup W \mid Z) \Rightarrow I(X, Y \mid Z)$ and $I(X, W \mid Z)$
  This relation asserts that if Z separates X from $Y \cup W$, then it also separates X from every subset of $Y \cup W$.
- *P3: Weak union*: $I(X, Y \cup W \mid Z) \Rightarrow I(X, W \mid Y \cup Z)$
  This relation asserts that if Z separates X from $Y \cup W$, then Z can be augmented by Y and still separate X from W.
- *P4: Contraction*: $I(X, W \mid Y \cup Z)$ and $I(X, Y \mid Z) \Rightarrow I(X, Y \cup W \mid Z)$
  This relation asserts that if $Y \cup Z$ separates X from W, then the separator $Y \cup Z$ can be reduced from the subset Y which will be added to W, if the remaining part i.e. Z, separates X from the deleted part Y.
- *P5: Intersection*: $I(X, Y \mid Z \cup W)$ and $I(X, W \mid Y \cup Z) \Rightarrow I(X, Y \cup W \mid Z)$
  This relation states that if within some set of variables $S = X \cup Y \cup Z \cup W$, X can be separated from the rest of S by two different subsets, $S1$ and $S2$ (i.e. $S1 = Y \cup Z$ and $S2 = Z \cup W$), then the intersection of $S1$ and $S2$ is sufficient to separate X from the rest of S.

Any independence structure that satisfies the properties P1-P4 is called a *semi-graphoid*. If it also satisfies property P5 it is said to be a *graphoid*. It has been shown that the probabilistic independence relation is a *semi-graphoid*, and it is a *graphoid* if the considered probability distribution is strictly positive (i.e. $p > 0$)[22].

Graphoid properties have been studied for several possibilistic independence relations. Indeed, Fonck[18] has shown that I_{NI} and I_{Prod} relations are *semi-graphoids*.

I_{NI} does not satisfy the intersection property, while I_{Prod} satisfies this property only if we consider strictly positive distributions. I_M independence relation satisfies all *graphoid* properties except the symmetry and its symmetrized version I_{MS} is a *graphoid*.

5. Graphoid properties of non-symmetric independence relations

5.1. *Reverse graphoid properties*

Graphoid properties are stated for symmetric relations while most of qualitative possibilistic independences are not naturally symmetric (the symmetry is generally enforced). For instance, the decomposition property shows how to derive $I(X, Y \mid Z)$ from $I(X, Y \cup W \mid Z)$. If the relation is naturally symmetric, then one also derives $I(X, Y \mid Z)$ from $I(Y \cup W, X \mid Z)$. However, if the relation is not symmetric, then there is no guarantee to derive $I(X, Y \mid Z)$ from $I(Y \cup W, X, \mid Z)$ even the relation satisfies the decomposition property. Thus, we also propose to study the symmetric counterparts of graphoid properties called *reverse graphoid properties*, which has been recently proposed by Vantaggi[25] when studying conditional independence in coherent conditional probabilistic framework:

- *Reverse-Decomposition*: $I(X \cup Y, W \mid Z) \Rightarrow I(Y, W \mid Z)$ and $I(X, W \mid Z)$
  This relation asserts that if W is irrelevant to $X \cup Y$ in the context of Z then W is irrelevant to Y (resp. X) in the same context.
- *Reverse-Weak union*: $I(X \cup Y, W \mid Z) \Rightarrow I(X, W \mid Y \cup Z)$
  This relation asserts that if Z makes W irrelevant to $X \cup Y$, then Z can be augmented by Y and still make W irrelevant to X.
- *Reverse-Contraction*: $I(X, W \mid Y \cup Z)$ and $I(Y, W \mid Z) \Rightarrow I(X \cup Y, W \mid Z)$
  This relation asserts that if $Y \cup Z$ separates X from W, then the separator $Y \cup Z$ can be reduced from the subset Y which will be added to X, if the remaining part i.e. Z, separates the deleted part Y from W.
- *Reverse-Intersection*:
  $I(Y, W \mid Z \cup X)$ and $I(X, W \mid Y \cup Z) \Rightarrow I(X \cup Y, W \mid Z)$
  This relation states that if within some set of variables $S = X \cup Y \cup Z \cup W$, W is irrelevant to the rest of S by two different subsets, $S1$ and $S2$ (i.e. $S1 = Z \cup X$ and $S2 = Z \cup Y$), then the intersection of $S1$ and $S2$ is sufficient to make irrelevant W from the rest of S.

It is important to note that if a symmetric relation satisfies any of the graphoid properties, then it satisfies its reverse counterpart too.

However, it may happen that, for a given plausibility relation, a non-symmetric independence relation (e.g., I_{BP}) fails to satisfy any of the reverse graphoid properties (e.g., reverse weak union), while its symmetrized version (e.g., I_{BPS}) satisfies such graphoid property (e.g., weak union).

The following subsections establish the graphoid properties of non-symmetric qualitative independence relations.

5.2. *Properties of belief-preserving independence*

Proposition 1. *I_{BP} satisfies all graphoid properties except the symmetry. Moreover, it satisfies the reverse-decomposition but it fails to satisfy the reverse-weak union, the reverse-contraction and the reverse-intersection.*

The proofs are reported in the appendix. Here, we only provide counter-examples.

Counter-example 1. LACK OF SYMMETRY PROPERTY FOR I_{BP}
Let us consider two binary variables A and B with the following plausibility relation: $a_1 \wedge b_1 >_\pi a_1 \wedge b_2 >_\pi a_2 \wedge b_2 >_\pi a_2 \wedge b_1$.

Table 1 shows that $I_{BP}(A, B \mid \emptyset)$ is true, namely $\forall a \in D_A, \forall b \in D_B$, $\mathbf{Acc}(a \mid b) = \mathbf{Acc}(a)$. However, $I_{BP}(B, A \mid \emptyset)$ is false, for instance $\mathbf{Acc}(b_1) = 1 \neq \mathbf{Acc}(b_1 \mid a_2) = -1$.

Table 1. Lack of symmetry property for I_{BP}

a	b	$Acc(a \mid b)$	$Acc(a)$	$Acc(b \mid a)$	$Acc(b)$
a_1	b_1	1	1	1	1
a_1	b_2	1	1	-1	-1
a_2	b_1	-1	-1	**-1**	**1**
a_2	b_2	-1	-1	1	-1

Counter-example 2. : LACK OF REVERSE-WEAK UNION PROPERTY FOR I_{BP}
Let us consider three binary variables A, B and C with the following plausibility relation: $a_1 \wedge b_1 \wedge c_1 >_\pi a_1 \wedge b_2 \wedge c_1 >_\pi a_1 \wedge b_1 \wedge c_2 >_\pi a_2 \wedge b_2 \wedge c_1 >_\pi a_1 \wedge b_2 \wedge c_2 =_\pi a_2 \wedge b_2 \wedge c_2 >_\pi a_2 \wedge b_1 \wedge c_1 >_\pi a_2 \wedge b_1 \wedge c_2$.

Table 2 shows that $I_{BP}(A \cup B, C \mid \emptyset)$ is true, namely, $\forall a \in D_A, \forall b \in D_B, \forall c \in D_C$, we have $\mathbf{Acc}(a \wedge b \mid c) = \mathbf{Acc}(a \wedge b)$.
However, $I_{BP}(A, C \mid B)$ is false since $\mathbf{Acc}(a_2 \mid b_2 \wedge c_2) = 0 \neq \mathbf{Acc}(a_2 \mid b_2) = -1$.

Table 2. Validity of $I_{BP}(A \cup B, C \mid \emptyset)$

a	b	$\mathbf{Acc}(a \wedge b \mid c_1)$	$\mathbf{Acc}(a \wedge b \mid c_2)$	$\mathbf{Acc}(a \wedge b)$
a_1	b_1	1	1	1
a_1	b_2	-1	-1	-1
a_2	b_1	-1	-1	-1
a_2	b_2	-1	-1	-1

Counter-example 3. : LACK OF REVERSE-CONTRACTION PROPERTY FOR I_{BP}
Let us consider three binary variables A, B and C with the following plausibility

relation: $a_1 \wedge b_1 \wedge c_1 >_\pi a_1 \wedge b_2 \wedge c_1 >_\pi a_1 \wedge b_2 \wedge c_2 >_\pi a_1 \wedge b_1 \wedge c_2 >_\pi a_2 \wedge b_1 \wedge c_1 >_\pi a_2 \wedge b_1 \wedge c_2 >_\pi a_2 \wedge b_2 \wedge c_1 >_\pi a_2 \wedge b_2 \wedge c_2$.

Tables 3 and 4, respectively, show that $I_{BP}(A, C \mid B)$ and $I_{BP}(A, C \mid \emptyset)$ are true, namely,
$\forall a \in D_A, \forall b \in D_B, \forall c \in D_C, \mathbf{Acc}(a \mid b \wedge c) = \mathbf{Acc}(a \mid b)$ and
$\forall a \in D_A, \forall c \in D_C, \mathbf{Acc}(a \mid c) = \mathbf{Acc}(a)$. However, $I_{BP}(A \cup B, C \mid \emptyset)$ is false since:
$\mathbf{Acc}(a_1 \wedge b_1 \mid c_2) = -1 \neq \mathbf{Acc}(a_1 \wedge b_1) = 1$.

Table 3. Validity of $I_{BP}(A, C \mid B)$

a	b	$\mathbf{Acc}(a \mid b \wedge c_1)$	$\mathbf{Acc}(a \mid b \wedge c_2)$	$\mathbf{Acc}(a \mid b)$
a_1	b_1	1	1	1
a_1	b_2	1	1	1
a_2	b_1	-1	-1	-1
a_2	b_2	-1	-1	-1

Table 4. Validity of $I_{BP}(A, C \mid \emptyset)$

a	c	$\mathbf{Acc}(a \mid c)$	$\mathbf{Acc}(a)$
a_1	c_1	1	1
a_1	c_2	1	1
a_2	c_1	-1	-1
a_2	c_2	-1	-1

Counter-example 4. : Lack of reverse-intersection property for I_{BP}
Let us consider three binary variables A, B and C with the following plausibility relation: $a_1 \wedge b_1 \wedge c_1 >_\pi a_2 \wedge b_2 \wedge c_1 >_\pi a_2 \wedge b_1 \wedge c_1 >_\pi a_1 \wedge b_1 \wedge c_2 =_\pi a_2 \wedge b_2 \wedge c_2 >_\pi a_1 \wedge b_2 \wedge c_1 >_\pi a_1 \wedge b_2 \wedge c_2 >_\pi a_2 \wedge b_1 \wedge c_2$.

Table 5 shows that $I_{BP}(B, C \mid A)$ and $I_{BP}(A, C \mid B)$ are true, namely, $\forall a \in D_A, \forall b \in D_B, \forall c \in D_C, \mathbf{Acc}(b \mid a \wedge c) = \mathbf{Acc}(b \mid a)$ and $\mathbf{Acc}(a \mid b \wedge c) = \mathbf{Acc}(a \mid b)$. However, $I_{BP}(A \cup B, C \mid \emptyset)$ is false since: $\mathbf{Acc}(a_2 \wedge b_2 \mid c_2) = 0 \neq \mathbf{Acc}(a_2 \wedge b_2) = -1$.

Table 5. Validity of $I_{BP}(B, C \mid A)$ and $I_{BP}(A, C \mid B)$

a	b	$\mathbf{Acc}(b \mid a \wedge c_1)$	$\mathbf{Acc}(b \mid a \wedge c_2)$	$\mathbf{Acc}(b \mid a)$
a_1	b_1	1	1	1
a_1	b_2	-1	-1	-1
a_2	b_1	-1	-1	-1
a_2	b_2	1	1	1
a	b	$\mathbf{Acc}(a \mid b \wedge c_1)$	$\mathbf{Acc}(a \mid b \wedge c_2)$	$\mathbf{Acc}(a \mid b)$
a_1	b_1	1	1	1
a_1	b_2	-1	-1	-1
a_2	b_1	-1	-1	-1
a_2	b_2	1	1	1

5.3. *Properties of preserving-ordering independence*

Proposition 2. *I_{PO} independence relation satisfies all graphoid properties except the symmetry. Moreover, it satisfies the reverse-decomposition and the reverse-weak union properties but neither the reverse-contraction nor the reverse-intersection are satisfied.*

Counter-example 5. Lack of symmetry property for I_{PO}
Let us consider two binary variables A and B with the following plausibility relation: $a_1 \wedge b_1 >_\pi a_1 \wedge b_2 >_\pi a_2 \wedge b_2 >_\pi a_2 \wedge b_1$.

- The local plausibility relation relative to A is $a_1 >_\Pi a_2$. Moreover, in the context b_1 (resp. b_2), we have $a_1 >_\Pi a_2$ since $a_1 \wedge b_1 >_\Pi a_2 \wedge b_1$ (resp. $a_1 \wedge b_2 >_\Pi a_2 \wedge b_2$). Thus, the relation $I_{PO}(A, B \mid \emptyset)$ is true since the ordering relative to the different instances of A is preserved for all instances of B.
- The local plausibility relation relative to B is $b_1 >_\Pi b_2$. However, in the context a_2, we have $b_2 >_\Pi b_1$, thus, the relation $I_{PO}(B, A \mid \emptyset)$ is false, since the ordering between b_1 and b_2 is not preserved in the context a_2.

Counter-example 6. : Lack of reverse-contraction property for I_{PO}
Let us consider three binary variables A, B and C with the following plausibility relation:
$a_2 \wedge b_2 \wedge c_1 >_\pi a_2 \wedge b_2 \wedge c_2 >_\pi a_2 \wedge b_1 \wedge c_1 >_\pi a_2 \wedge b_1 \wedge c_2 >_\pi a_1 \wedge b_2 \wedge c_1 >_\pi a_1 \wedge b_1 \wedge c_1 >_\pi a_1 \wedge b_2 \wedge c_2 >_\pi a_1 \wedge b_1 \wedge c_2$.

Let us check that $I_{PO}(C, A \mid B)$ and $I_{PO}(B, A \mid \emptyset)$ are indeed satisfied while $I_{PO}(B \cup C, A \mid \emptyset)$ is not satisfied.

- In the context of b_1 (resp. b_2), the local plausibility relation relative to C is $c_1 >_\Pi c_2$ (resp. $c_1 >_\Pi c_2$). This order is preserved after the revision by a_1 and a_2 since $a_1 \wedge b_1 \wedge c_1 >_\pi a_1 \wedge b_1 \wedge c_2$ and $a_2 \wedge b_1 \wedge c_1 >_\pi a_2 \wedge b_1 \wedge c_2$ (resp. $a_1 \wedge b_2 \wedge c_1 >_\pi a_1 \wedge b_2 \wedge c_2$ and $a_2 \wedge b_2 \wedge c_1 >_\pi a_2 \wedge b_2 \wedge c_2$). Thus, the relation $I_{PO}(C, A \mid B)$ is true.
- The local plausibility relation relative to B is $b_2 >_\Pi b_1$. Moreover, in the context a_1 (resp. a_2), we have $b_2 >_\Pi b_1$ since $a_1 \wedge b_2 >_\Pi a_1 \wedge b_1$ (resp. $a_2 \wedge b_2 >_\Pi a_2 \wedge b_1$). Thus, the relation $I_{PO}(B, A \mid \emptyset)$ is true since the ordering relative to the different instances of B is preserved for all instances of A.
- The local plausibility relation relative to $B \cup C$ is $b_2 \wedge c_1 >_\Pi b_2 \wedge c_2 >_\Pi b_1 \wedge c_1 >_\Pi b_1 \wedge c_2$. However, in the context a_1 we have $b_1 \wedge c_1 >_\Pi b_2 \wedge c_2$. Thus, the order is not preserved and the relation $I_{PO}(B \cup C, A \mid \emptyset)$ is false.

Counter-example 7. : Lack of reverse-intersection property for I_{PO}
Let us consider again the plausibility relation given in Counter-example 6, where we already checked that $I_{PO}(C, A \mid B)$ is true and $I_{PO}(B \cup C, A \mid \emptyset)$ is false. Let us check that $I_{PO}(B, A \mid C)$ is true too.

This relation is true. Indeed, in the context of c_1 (resp. c_2), the local plausibility relation relative to B is $b_2 >_\Pi b_1$ (resp. $b_2 >_\Pi b_1$). This order is preserved after the revision by a_1 and a_2 since $a_1 \wedge b_2 \wedge c_1 >_\pi a_1 \wedge b_1 \wedge c_1$ and $a_2 \wedge b_2 \wedge c_1 >_\pi a_2 \wedge b_1 \wedge c_1$ (resp. $a_1 \wedge b_2 \wedge c_2 >_\pi a_1 \wedge b_1 \wedge c_2$ and $a_2 \wedge b_2 \wedge c_2 >_\pi a_2 \wedge b_1 \wedge c_2$).

6. Graphoid properties of symmetric independence relations

This section establishes the graphoid properties of symmetric or symmetrized qualitative independence relations.

6.1. *Properties of symmetrized belief-preserving independence*

Proposition 3.
I_{BPS} satisfies the symmetry (by definition), the decomposition but it fails to satisfy the weak union, the contraction and the intersection.

The proof of decomposition property is immediate since I_{BP} satisfies the decomposition and the reverse-decomposition properties.
Counter-example 8. : Lack of weak union property for I_{BPS}
Let us consider again the plausibility relation given in Counter-example 2. In this plausibility relation $I_{BPS}(A, C \mid B)$ is false since $I_{BP}(A, C \mid B)$ is false.

Moreover, we have checked that $I_{BP}(A \cup B, C \mid \emptyset)$ is true. Thus, it is enough to check $I_{BP}(C, A \cup B \mid \emptyset)$ to establish $I_{BPS}(A \cup B, C \mid \emptyset)$ an hence to falsify the weak union property. Table 6 shows that this relation is indeed true. Namely, $\forall a \in D_A, \forall b \in D_B, \forall c \in D_C$, we have $\mathbf{Acc}(c \mid a \wedge b) = \mathbf{Acc}(c)$.

Table 6. Validity of $I_{BP}(C, A \cup B \mid \emptyset)$

a	b	$\mathbf{Acc}(c_1 \mid a \wedge b)$	$\mathbf{Acc}(c_1)$	$\mathbf{Acc}(c_2 \mid a \wedge b)$	$\mathbf{Acc}(c_2)$
a_1	b_1	1	1	-1	-1
a_1	b_2	1	1	-1	-1
a_2	b_1	1	1	-1	-1
a_2	b_2	1	1	-1	-1

Counter-example 9. : Lack of contraction property for I_{BPS}

Let us consider again the plausibility relation given in Counter-example 3. In this plausibility relation $I_{BPS}(A \cup B, C \mid \emptyset)$ is false since $I_{BP}(A \cup B, C \mid \emptyset)$ is false. Moreover, we have checked that $I_{BP}(A, C \mid B)$ and $I_{BP}(A, C \mid \emptyset)$ are true. Thus, it is enough to check $I_{BP}(C, A \mid B)$ and $I_{BP}(C, A \mid \emptyset)$ to establish $I_{BPS}(A, C \mid B)$ and $I_{BPS}(A, C \mid \emptyset)$ and hence to falsify the contraction property. Tables 7 and 8 show, respectively, that these two relations are indeed true.

Namely, $\forall a \in D_A, \forall b \in D_B, \forall c \in D_C, \mathbf{Acc}(c \mid a \wedge b) = \mathbf{Acc}(c \mid b)$ and $\forall a \in D_A, \forall c \in D_C, \mathbf{Acc}(c \mid a) = \mathbf{Acc}(c)$.

Table 7. Validity of $I_{BP}(C, A \mid B)$

c	b	$\mathbf{Acc}(c \mid a_1 \wedge b)$	$\mathbf{Acc}(c \mid a_2 \wedge b)$	$\mathbf{Acc}(c \mid b)$
c_1	b_1	1	1	1
c_1	b_2	-1	-1	-1
c_2	b_1	1	1	1
c_2	b_2	-1	-1	-1

Table 8. Validity of $I_{BP}(C, A \mid \emptyset)$

a	c	$\mathbf{Acc}(c \mid a)$	$\mathbf{Acc}(c)$
a_1	c_1	1	1
a_1	c_2	-1	-1
a_2	c_1	1	1
a_2	c_2	-1	-1

Counter-example 10. : Lack of intersection property for I_{BPS}
Let us consider again the plausibility relation given in Counter-example 4. In this plausibility relation $I_{BPS}(A \cup B, C \mid \emptyset)$ is false since $I_{BP}(A \cup B, C \mid \emptyset)$ is false. Moreover, we have checked that $I_{BP}(B, C \mid A)$ and $I_{BP}(A, C \mid B)$ are true. Thus, it is enough to check $I_{BP}(C, B \mid A)$ and $I_{BP}(C, A \mid B)$ to establish $I_{BPS}(B, C \mid A)$ and $I_{BPS}(A, C \mid B)$. Table 9 shows that these relations are indeed true.
Namely, $\forall a \in D_A, \forall b \in D_B, \forall c \in D_C, \mathbf{Acc}(c \mid a \wedge b) = \mathbf{Acc}(c \mid a)$ and
$\forall a \in D_A, \forall c \in D_C, \mathbf{Acc}(c \mid a \wedge b) = \mathbf{Acc}(c \mid b)$.

Table 9. Validity of $I_{BP}(C, B \mid A)$ and $I_{BP}(C, A \mid B)$

a	c	$\mathbf{Acc}(c \mid a \wedge b_1)$	$\mathbf{Acc}(c \mid a \wedge b_2)$	$\mathbf{Acc}(c \mid a)$
a_1	c_1	1	1	1
a_1	c_2	-1	-1	-1
a_2	c_1	1	1	1
a_2	c_2	-1	-1	-1
b	c	$\mathbf{Acc}(c \mid a_1 \wedge b)$	$\mathbf{Acc}(c \mid a_1 \wedge b)$	$\mathbf{Acc}(c \mid b)$
b_1	c_1	1	1	1
b_1	c_2	-1	-1	-1
b_2	c_1	1	1	1
b_2	c_2	-1	-1	-1

6.2. *Properties of symmetrized preserving-ordering independence*

Proposition 4. *I_{POS} satisfies the symmetry (by definition), the decomposition and the weak union but neither the contraction nor the intersection.*

The proof of decomposition (resp. weak union) property is immediate since I_{PO} satisfies the decomposition (resp. weak union) and the reverse-decomposition (resp. reverse-weak union) properties.

Counter-example 11. : Lack of contraction property for I_{POS}
Let us consider again the plausibility relation given in Counter-example 6. We have already checked that $I_{PO}(B \cup C, A \mid \emptyset)$ is false which implies that $I_{POS}(B \cup C, A \mid \emptyset)$ is false too. Moreover, we have checked that $I_{PO}(C, A \mid B)$ and $I_{PO}(B, A \mid \emptyset)$ are true. Thus, it is enough to test $I_{PO}(A, C \mid B)$ and $I_{PO}(A, B \mid \emptyset)$ to establish $I_{POS}(C, A \mid B)$ and $I_{POS}(B, A \mid \emptyset)$.

- In the context of b_1 (resp. b_2), the local plausibility relation relative to A is $a_2 >_\Pi a_1$ (resp. $a_2 >_\Pi a_1$). This order is preserved after the revision by c_1 and c_2 since $a_2 \wedge b_1 \wedge c_1 >_\pi a_1 \wedge b_1 \wedge c_1$ and $a_2 \wedge b_1 \wedge c_1 >_\pi a_1 \wedge b_1 \wedge c_1$ (resp. $a_2 \wedge b_2 \wedge c_1 >_\pi a_1 \wedge b_2 \wedge c_1$ and $a_2 \wedge b_2 \wedge c_1 >_\pi a_1 \wedge b_2 \wedge c_1$). Thus, the relation $I_{PO}(A, C \mid B)$ is true.

- The local plausibility relation relative to A is $a_2 >_\Pi a_1$. Moreover, in the context b_1 (resp. b_2), we have $a_2 >_\Pi a_1$ since $a_2 \wedge b_1 >_\Pi a_1 \wedge b_1$ (resp. $a_2 \wedge b_2 >_\Pi a_1 \wedge b_2$). Thus, the relation $I_{PO}(A, B \mid \emptyset)$ is true since the ordering relative to the different instances of A is preserved for all instances of B.

Counter-example 12. : Lack of intersection property for I_{POS}
Let us consider again the plausibility relation given in Counter-example 6.
In Counter-example 11 we have checked that $I_{POS}(C, A \mid B)$ is true and $I_{POS}(B \cup C, A \mid \emptyset)$ is false.

Moreover, in Counter-example 7 we have checked that $I_{PO}(B, A \mid C)$ is true, thus it is enough to check that $I_{PO}(A, B \mid C)$ is true to establish $I_{POS}(B, A \mid C)$.

This relation is true, indeed, in the context of c_1 (resp. c_2), the local plausibility relation relative to A is $a_2 >_\Pi a_1$ (resp. $a_2 >_\Pi a_1$). This order is preserved after the revision by b_1 and b_2 since $a_2 \wedge b_1 \wedge c_1 >_\pi a_1 \wedge b_1 \wedge c_1$ and $a_2 \wedge b_2 \wedge c_1 >_\pi a_1 \wedge b_2 \wedge c_1$ (resp. $a_2 \wedge b_1 \wedge c_2 >_\pi a_1 \wedge b_1 \wedge c_2$ and $a_2 \wedge b_2 \wedge c_2 >_\pi a_1 \wedge b_2 \wedge c_2$).

6.3. *Properties of belief decompositional independence*

Proposition 5. *I_{PT} relation is not a semi-graphoid, since it satisfies the symmetry, the decomposition and the contraction but neither the weak union nor the intersection properties.*

Counter-example 13. : Lack of weak union property for I_{PT}
Let us consider three binary variables A, B and C with the following plausibility relation: $a_2 \wedge b_2 \wedge c_1 >_\pi a_1 \wedge b_1 \wedge c_1 =_\pi a_2 \wedge b_1 \wedge c_2 >_\pi a_1 \wedge b_1 \wedge c_2 =_\pi a_2 \wedge b_1 \wedge c_1 >_\pi a_1 \wedge b_2 \wedge c_1 =_\pi a_1 \wedge b_2 \wedge c_2 =_\pi a_2 \wedge b_2 \wedge c_2$.

Table 10 shows that $I_{PT}(A, B \cup C \mid \emptyset)$ is true, namely,
$\forall a \in D_A, \forall b \in D_B, \forall c \in D_C, \mathbf{Acc}(a \wedge b \wedge c) = min(\mathbf{Acc}(a), \mathbf{Acc}(b \wedge c))$.
However, $I_{PT}(A, C \mid B)$ is false since:
$\mathbf{Acc}(a_1 \wedge c_2 \mid b_1) = -1 \neq min(\mathbf{Acc}(a_1 \mid b_1), \mathbf{Acc}(c_2 \mid b_1)) = 0$.

Table 10. Validity of $I_{PT}(A, B \cup C \mid \emptyset)$

a b c	$\mathbf{Acc}(a \wedge b \wedge c)$	$\mathbf{Acc}(a)$	$\mathbf{Acc}(b \wedge c)$	a b c	$\mathbf{Acc}(a \wedge b \wedge c)$	$\mathbf{Acc}(a)$	$\mathbf{Acc}(b \wedge c)$
$a_1\, b_1\, c_1$	-1	-1	-1	$a_2\, b_1\, c_1$	-1	1	-1
$a_1\, b_1\, c_2$	-1	-1	-1	$a_2\, b_1\, c_2$	-1	1	-1
$a_1\, b_2\, c_1$	-1	-1	1	$a_2\, b_2\, c_1$	1	1	1
$a_1\, b_2\, c_2$	-1	-1	-1	$a_2\, b_2\, c_2$	-1	1	-1

Counter-example 14. : Lack of intersection property for I_{PT}
Let us consider three binary variables A, B and C with the following plausibility relation: $a_1 \wedge b_2 \wedge c_2 =_\pi a_2 \wedge b_1 \wedge c_1 >_\pi a_1 \wedge b_1 \wedge c_1 =_\pi a_1 \wedge b_1 \wedge c_2 =_\pi a_1 \wedge b_2 \wedge c_1 =_\pi a_2 \wedge b_1 \wedge c_2 =_\pi a_2 \wedge b_2 \wedge c_1 =_\pi a_2 \wedge b_2 \wedge c_2$.

Table 11 shows that $I_{PT}(A, B \mid C)$ and $I_{PT}(A, C \mid B)$ are true, namely
$\forall a \in D_A, \forall b \in D_B, \forall c \in D_C$,
$\mathbf{Acc}(a \wedge b \mid c) = min(\mathbf{Acc}(a \mid c), \mathbf{Acc}(b \mid c))$ and
$\mathbf{Acc}(a \wedge c \mid b) = min(\mathbf{Acc}(a \mid b), \mathbf{Acc}(c \mid b))$.
However, $I_{PT}(A, B \cup C \mid \emptyset)$ is false since:
$\mathbf{Acc}(a_2 \wedge b_2 \wedge c_2) = -1 \neq min(\mathbf{Acc}(a_2), \mathbf{Acc}(b_2 \wedge c_2)) = 0$.

Table 11. Validity of $I_{PT}(A, B \mid C)$ and $I_{PT}(A, C \mid B)$

a b	$\mathbf{Acc}(a \wedge b \mid c_1)$	$\mathbf{Acc}(a \mid c_1)$	$\mathbf{Acc}(b \mid c_1)$	$\mathbf{Acc}(a \wedge b \mid c_2)$	$\mathbf{Acc}(a \mid c_2)$	$\mathbf{Acc}(b \mid c_2)$
a_1 b_1	-1	-1	1	-1	1	-1
a_1 b_2	-1	-1	-1	1	1	1
a_2 b_1	1	1	1	-1	-1	-1
a_2 b_2	-1	1	-1	-1	-1	1
a c	$\mathbf{Acc}(a \wedge c \mid b_1)$	$\mathbf{Acc}(a \mid b_1)$	$\mathbf{Acc}(c \mid b_1)$	$\mathbf{Acc}(a \wedge c \mid b_2)$	$\mathbf{Acc}(a \mid b_2)$	$\mathbf{Acc}(c \mid b_2)$
a_1 c_1	-1	-1	1	-1	1	-1
a_1 c_2	-1	-1	-1	1	1	1
a_2 c_1	1	1	1	-1	-1	-1
a_2 c_2	-1	1	-1	-1	-1	1

6.4. *Properties of decompositional independence based on remarkable plausibility relations*

Proposition 6. *I_{Pareto} independence is a* graphoid.

The proof of this proposition is immediate since I_{Pareto} is equivalent to I_{MS} independence relation[2] which is a graphoid.

Proposition 7.
$I_{leximax}$ and $I_{leximin}$ only satisfy the symmetry and the decomposition and fail to satisfy weak union, contraction and intersection properties.

Some properties may be recovered in particular cases. For instance in the case of of binary variables and two-level distributions, $\mathrm{I}_{leximax}$ and $\mathrm{I}_{leximin}$ relations satisfy the weak union since they are equivalent to I_{POS}[2].

Counter-example 15. : LACK OF WEAK UNION PROPERTY FOR $I_{leximax}$
Let us consider three variables A, B and C with the following plausibility relation:
$a_1 \wedge b_1 \wedge c_1 >_\pi a_2 \wedge b_1 \wedge c_1 >_\pi a_1 \wedge b_2 \wedge c_2 >_\pi a_3 \wedge b_1 \wedge c_1 =_\pi a_1 \wedge b_1 \wedge c_2 >_\pi a_1 \wedge b_2 \wedge c_1 >_\pi a_2 \wedge b_2 \wedge c_2 >_\pi a_2 \wedge b_1 \wedge c_2 >_\pi a_2 \wedge b_2 \wedge c_1 >_\pi a_3 \wedge b_2 \wedge c_2 >_\pi a_3 \wedge b_1 \wedge c_2 >_\pi a_3 \wedge b_2 \wedge c_1$.
It can be checked that $I_{leximax}(A, B \cup C \mid \emptyset)$ is true since $a_1 =_\Pi b_1 \wedge c_1 >_\Pi a_2 >_\Pi a_3 =_\Pi b_1 \wedge c_2 >_\Pi b_2 \wedge c_1 >_\Pi b_2 \wedge c_2$. However, $I_{leximax}(A, C \mid B)$ is false since $a_2 \wedge b_2 \wedge c_1 >_\Pi a_3 \wedge b_2 \wedge c_2$ while $max(a_3 \wedge b_2, b_2 \wedge c_2) >_\Pi max(a_2 \wedge b_2, b_2 \wedge c_1)$.

Counter-example 16. : LACK OF CONTRACTION PROPERTY FOR $I_{leximax}$
Let us consider three binary variables A, B and C with the following plausibility relation: $a_1 \wedge b_2 \wedge c_1 >_\pi a_1 \wedge b_2 \wedge c_2 =_\pi a_2 \wedge b_2 \wedge c_1 >_\pi a_2 \wedge b_2 \wedge c_2 >_\pi a_1 \wedge b_1 \wedge c_2 >_\pi a_1 \wedge b_1 \wedge c_1 =_\pi a_2 \wedge b_1 \wedge c_2 >_\pi a_2 \wedge b_1 \wedge c_1$.
It can be checked that $I_{leximax}(A, B \mid \emptyset)$ and $I_{leximax}(A, C \mid B)$ are true since $a_1 =_\Pi b_2 >_\Pi a_2 >_\Pi b_1$ and $a_1 \wedge b_2 =_\Pi b_2 \wedge c_1 >_\Pi a_2 \wedge b_2 =_\Pi b_2 \wedge c_2 >_\Pi a_1 \wedge b_1 =_\Pi b_1 \wedge c_2 >_\Pi a_2 \wedge b_1 =_\Pi b_1 \wedge c_1$. However, $I_{leximax}(A, B \cup C \mid \emptyset)$ is false since $a_1 \wedge b_1 \wedge c_1 =_\pi a_2 \wedge b_1 \wedge c_2$ while $max(a_1, b_1 \wedge c_1) >_\Pi max(a_2, b_1 \wedge c_2)$.

Counter-example 17. : LACK OF INTERSECTION PROPERTY FOR $I_{leximax}$
Let us consider three binary variables A, B and C with the following plausibility relation: $a_1 \wedge b_1 \wedge c_1 =_\pi a_1 \wedge b_2 \wedge c_2 >_\pi a_1 \wedge b_2 \wedge c_1 >_\pi a_1 \wedge b_1 \wedge c_2 >_\pi a_2 \wedge b_2 \wedge c_2 >_\pi a_2 \wedge b_1 \wedge c_1 >_\pi a_2 \wedge b_2 \wedge c_1 >_\pi a_2 \wedge b_1 \wedge c_2$.
It can be checked that $I_{leximax}(A, B \mid C)$ and $I_{leximax}(A, C \mid B)$ are true since $a_1 \wedge c_1 =_\Pi a_1 \wedge c_2 =_\Pi b_1 \wedge c_1 =_\Pi b_2 \wedge c_2 >_\Pi b_2 \wedge c_1 >_\Pi b_1 \wedge c_2 >_\Pi a_2 \wedge c_2 >_\Pi a_2 \wedge c_1$ and $a_1 \wedge b_1 =_\Pi a_1 \wedge b_2 =_\Pi b_1 \wedge c_1 =_\Pi b_2 \wedge c_2 >_\Pi b_2 \wedge c_1 >_\Pi b_1 \wedge c_2 >_\Pi a_2 \wedge b_2 >_\Pi a_2 \wedge b_1$. However, $I_{leximax}(A, B \cup C \mid \emptyset)$ is false since $a_2 \wedge b_2 \wedge c_2 >_\Pi a_2 \wedge b_1 \wedge c_1$ while $max(a_2, b_2 \wedge c_2) =_\Pi max(a_2, b_1 \wedge c_1)$ and $min(a_2, b_2 \wedge c_2) =_\Pi min(a_2, b_1 \wedge c_1)$.

Counter-example 18. : LACK OF WEAK UNION PROPERTY FOR $I_{leximin}$
Let us consider three variables A, B and C with the following plausibility relation: $a_3 \wedge b_2 \wedge c_3 >_\pi a_2 \wedge b_2 \wedge c_3 =_\pi a_3 \wedge b_1 \wedge c_1 >_\pi a_2 \wedge b_1 \wedge c_1 >_\pi a_1 \wedge b_2 \wedge c_3 =_\pi a_3 \wedge b_1 \wedge c_2 >_\pi a_1 \wedge b_1 \wedge c_1 =_\pi a_2 \wedge b_1 \wedge c_2 >_\pi a_1 \wedge b_1 \wedge c_2 >_\pi a_3 \wedge b_1 \wedge c_3 >_\pi a_2 \wedge b_1 \wedge c_3 >_\pi a_1 \wedge b_1 \wedge c_3 >_\pi a_3 \wedge b_2 \wedge c_1 >_\pi a_2 \wedge b_2 \wedge c_1 >_\pi a_1 \wedge b_2 \wedge c_1 >_\pi a_3 \wedge b_2 \wedge c_2 >_\pi a_2 \wedge b_2 \wedge c_2 >_\pi a_1 \wedge b_2 \wedge c_2$
It can be checked that $I_{leximin}(A, B \cup C \mid \emptyset)$ is true since $a_3 =_\Pi b_1 \wedge c_3 >_\Pi a_2 =_\Pi b_1 \wedge c_1 >_\Pi a_1 =_\Pi b_1 \wedge c_2 >_\Pi b_1 \wedge c_3 >_\Pi b_2 \wedge c_1 >_\Pi b_2 \wedge c_2$.
However, $I_{leximin}(A, C \mid B)$ is false since $a_1 \wedge b_1 \wedge c_1 =_\Pi a_2 \wedge b_1 \wedge c_2$ while $min(a_1 \wedge b_1, b_1 \wedge c_1) <_\Pi min(a_2 \wedge b_1, b_1 \wedge c_2)$.

Counter-example 19. : LACK OF CONTRACTION PROPERTY FOR $I_{leximin}$
Let us consider three binary variables A, B and C with the following plausibility relation: $a_1 \wedge b_2 \wedge c_1 >_\pi a_1 \wedge b_2 \wedge c_2 =_\pi a_2 \wedge b_2 \wedge c_1 >_\pi a_2 \wedge b_2 \wedge c_2 >_\pi a_1 \wedge b_1 \wedge c_2 >_\pi a_1 \wedge b_1 \wedge c_1 =_\pi a_2 \wedge b_1 \wedge c_2 >_\pi a_2 \wedge b_1 \wedge c_1$.
It can be checked that $I_{leximin}(A, B \mid \emptyset)$ and $I_{leximin}(A, C \mid B)$ are true since $a_1 =_\Pi b_2 >_\Pi a_2 >_\Pi b_1$ and $a_1 \wedge b_2 =_\Pi b_2 \wedge c_1 >_\Pi a_2 \wedge b_2 =_\Pi b_2 \wedge c_2 >_\Pi a_1 \wedge b_1 =_\Pi b_1 \wedge c_2 >_\Pi a_2 \wedge b_1 =_\Pi b_1 \wedge c_1$. However, $I_{leximin}(A, B \cup C \mid \emptyset)$ is false since $a_1 \wedge b_1 \wedge c_1 =_\pi a_2 \wedge b_1 \wedge c_2$ while $min(a_2, b_1 \wedge c_2) >_\Pi min(a_1, b_1 \wedge c_1)$.

Counter-example 20. : LACK OF INTERSECTION PROPERTY FOR $I_{leximin}$
Let us consider the plausibility relation given in the previous example. It can be checked that $I_{leximin}(A, B \mid C)$ and $I_{leximin}(A, C \mid B)$ are true since $a_1 \wedge c_1 =_\Pi b_2 \wedge c_1 >_\Pi a_1 \wedge c_2 =_\Pi a_2 \wedge c_1 =_\Pi b_2 \wedge c_2 >_\Pi a_2 \wedge c_2 >_\Pi b_1 \wedge c_2 >_\Pi b_1 \wedge c_1$ and $a_1 \wedge b_2 =_\Pi b_2 \wedge c_1 >_\Pi a_2 \wedge b_2 =_\Pi b_2 \wedge c_2 >_\Pi a_1 \wedge b_1 =_\Pi b_1 \wedge c_2 >_\Pi a_2 \wedge b_1 =_\Pi b_1 \wedge c_1$. However, $I_{leximin}(A, B \cup C \mid \emptyset)$ is false since $a_1 \wedge b_1 \wedge c_1 =_\pi a_2 \wedge b_1 \wedge c_2$ while $min(a_2, b_1 \wedge c_2) >_\Pi min(a_1, b_1 \wedge c_1)$.

7. Summary of graphoid properties

Table 12 summarizes results on graphoid properties and their reverse counterparts.

Table 12. Summary of graphoid properties

	Symmetry	Decomposition / R-decomposition	Weak union / R-weak union	Contracaction / R-contraction	Intersection / R-intersection
I_{BP}	no	yes/yes	yes/no	yes/no	yes/no
I_{BPS}	yes	yes	no	no	no
I_{PO}	no	yes/yes	yes/yes	yes/no	yes/no
I_{POS}	yes	yes	yes	no	no
I_{PT}	yes	yes	no	yes	no
$I_{leximax}$	yes	yes	no	no	no
$I_{leximin}$	yes	yes	no	no	no
I_{Pareto}	yes	yes	yes	yes	yes

Note that I_{BP} and I_{PO} have good properties since they satisfy all graphoid properties except the symmetry. Unfortunately, the addition of this property to I_{BP} leads to the loss of the weak union, contraction and intersection properties. In the same manner it leads to the loss of the contraction and intersection properties of I_{PO}. In addition, I_{Pareto} has good properties since it is a graphoid but is too strong[2] to be practically used.

8. Conclusion

In this paper, we have studied graphoid properties of qualitative possibilistic independence relations that we have proposed in[2].

Two kinds of independence have been investigated: *causal* and *decompositional* ones. Causal independence relations can be simply defined using notions of accepted, ignored and rejected beliefs. Decompositional independence relations are defined using other operators different from the traditional *minimum* and *product* operators such that the *leximin* and *leximax* operators.

Since, several of these qualitative possibilistic independence fails to satisfy the symmetry property, we have also proposed to analyze these non-symmetric relations with respect to the symmetric counterparts of graphoid properties called *reverse graphoid properties* (see[25]) . We have shown that adding the symmetry property can lead to the loose of some graphoid properties. For instance, adding the symmetry to the PO-independence causes the failure of the contraction and the intersection properties.

Note that similar behaviour appears with possibilistic independence based on conditional events proposed by Bouchon-Meunier et al.[4]. Indeed, adding the symmetry property to $I_{CE}(X, Y \mid Z)$ leads to the lost of weak union property.

Our study shows that except Pareto independence relation, there is no qualitative independence relation (symmetrized or not) which satisfies all graphoid properties. Moreover, only the decomposition property is satisfied by all these independence relations. This may be explained by the absence of commensurability assumption between the different orderings in the qualitative setting since we only use total pre-orders between events which are weaker than the common finite scale $[0, 1]$ used in the min and product based independence relations which have good graphoid properties.

Results on independence relations can be used for defining new forms of qualitative networks. For instance, Brafmann and col. [6] have proposed a new qualitative network where inside each node a plausibility relation is used instead of possibility degrees. They use *Ceteris Paribus* independence which is equivalent to the qualitative independence relation based on preserving orderings[2] (i.e. POS-independence). Therefore, our study of graphoid properties can be useful for showing the coherence of propagation algorithms based on Ceteris Paribus independence.

More generally, some care should be taken if one would like to develop local algorithms based on qualitative possibilistic independence. For instance, the simplifications, based on d-separation, used in local propagation algorithms in graphical models are not valid, and therefore, other conditions should be considered.

For example, Vantggi[25], in studying conditional independence in coherent conditions, has proposed a new separation criterion (called t-separation) for directed acyclic graphs which is appropriate for independence relations which do not satisfy the symmetry property.

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Appendix A.

We first give two technical lemmas which will be needed in some proofs (proofs of these lemmas can be found in[2]).

Lemma 1. *Let X, Y, Z be three mutually disjoint subsets of variables of V, then* $\forall x \in D_X, \forall y \in D_Y, \forall z \in D_Z$:
$\boldsymbol{Acc}(x \wedge y \mid z) \neq min(\boldsymbol{Acc}(x \mid z), \boldsymbol{Acc}(y \mid z))$
$\Leftrightarrow \boldsymbol{Acc}(x \wedge y \mid z) = -1, \boldsymbol{Acc}(x \mid z) = 0,$ *and* $\boldsymbol{Acc}(y \mid z) = 0$.

Lemma 2. *Let* $x \in D_X, \forall y \in D_Y, \forall z \in D_Z$. *Then:*
if $\boldsymbol{Acc}(x \wedge y \mid z) = 1$ *then* $\boldsymbol{Acc}(x \mid z) = 1$ *(resp.* $\boldsymbol{Acc}(y \mid z) = 1$).
if $\boldsymbol{Acc}(x \wedge y \mid z) = 0$ *then* $\boldsymbol{Acc}(x \mid z) \geq 0$. *(resp.* $\boldsymbol{Acc}(y \mid z) \geq 0$*).*

Proof of Proposition 1

- Decomposition property for I_{BP}.

We want to prove that $I_{BP}(X, Y \cup W \mid Z) \Rightarrow I_{BP}(X, Y \mid Z)$ and $I_{BP}(X, W \mid Z)$.
We only prove that $I_{BP}(X, Y \cup W \mid Z) \Rightarrow I_{BP}(X, Y \mid Z)$
(the proof of $I_{BP}(X, Y \cup W \mid Z) \Rightarrow I_{BP}(X, W \mid Z)$ is analogous).
This means that we want to prove that:
if (i) $\forall x \in D_X, \forall y \in D_Y, \forall z \in D_Z, \forall w \in D_W$, $\mathbf{Acc}(x \mid y \wedge z \wedge w) = \mathbf{Acc}(x \mid z)$
then (ii) $\forall x \in D_X, \forall y \in D_Y, \forall z \in D_Z, \forall w \in D_W$, $\mathbf{Acc}(x \mid y \wedge z) = \mathbf{Acc}(x \mid z)$.
Suppose that this implication is false, this means that (i) is true and (ii) is false.
This means that, $\exists x' \in D_X, y' \in D_Y, z' \in D_Z$, s.t $Acc(x' \mid y' \wedge z') \neq Acc(x' \mid z')$
Let us analyze the possible values for $\mathbf{Acc}(x' \mid z')$:

- $\mathbf{Acc}(x' \mid z') = 0$
  $\Rightarrow \exists x" \neq_\Pi x' \in D_X$ s.t. $\mathbf{Acc}(x" \mid z') = 0$
  $\Rightarrow \forall y \in D_Y, \forall w \in D_W, \mathbf{Acc}(x' \mid y \wedge z' \wedge w) = 0$ and $\mathbf{Acc}(x" \mid y \wedge z' \wedge w) = 0$ (from (i))
  $\Rightarrow \forall y \in D_Y, \forall w \in D_W, x' \wedge y \wedge z' \wedge w =_\Pi x" \wedge y \wedge z' \wedge w$.
  $\Rightarrow \forall y \in D_Y, max_w\{x' \wedge y \wedge z' \wedge w\} =_\Pi max_w\{x" \wedge y \wedge z' \wedge w\}$,
  $\Rightarrow \forall y \in D_Y, x' \wedge y \wedge z' =_\Pi x" \wedge y \wedge z'$.
  $\Rightarrow \forall y \in D_Y, \mathbf{Acc}(x' \mid y \wedge z') = 0$.
  $\Rightarrow \mathbf{Acc}(x' \mid y' \wedge z') = 0$ (when Y takes y' as particular value)
  Hence contradiction.
- $\mathbf{Acc}(x' \mid z') = -1$
  $\Rightarrow \forall y \in D_Y, \forall w \in D_W, \mathbf{Acc}(x' \mid y \wedge z' \wedge w) = -1$ (from (i))
  $\Rightarrow \forall y \in D_Y, \forall w \in D_W, \exists x" \neq_\Pi x' \in D_X$ s.t. $x" \wedge y \wedge z' \wedge w >_\Pi x' \wedge y \wedge z' \wedge w$
  $\Rightarrow \forall y \in D_Y, \exists x" \neq_\Pi x' \in D_X, \exists w" \in D_W$ s.t $\forall w \in D_W$,
  $x" \wedge y \wedge z' \wedge w" >_\Pi x' \wedge y \wedge z' \wedge w$
  $\Rightarrow \forall y \in D_Y, \exists x" \neq_\Pi x' \in D_X, \exists w" \in D_W$ s.t
  $x" \wedge y \wedge z' \wedge w" >_\Pi max_w\{x' \wedge y \wedge z' \wedge w\}$
  $\Rightarrow \forall y \in D_Y, \exists x" \neq_\Pi x' \in D_X, \exists w" \in D_W$ s.t $x" \wedge y \wedge z' \wedge w" >_\Pi x' \wedge y \wedge z'$
  $\Rightarrow \forall y \in D_Y, \exists x" \neq_\Pi x' \in D_X$ s.t $x" \wedge y \wedge z' >_\Pi x' \wedge y \wedge z'$
  (since $x" \wedge y \wedge z' \geq_\Pi x" \wedge y \wedge z' \wedge w"$)
  $\Rightarrow \forall y \in D_Y, \mathbf{Acc}(x' \mid y \wedge z') = -1$.
  $\Rightarrow \mathbf{Acc}(x' \mid y' \wedge z') = -1$ (when Y takes y' as particular value)
  Hence contradiction.
- $\mathbf{Acc}(x' \mid z') = 1$
  $\Rightarrow \forall y \in D_Y, \forall w \in D_W, \mathbf{Acc}(x' \mid y \wedge z' \wedge w) = 1$ (from (i))
  $\Rightarrow \forall y \in D_Y, \forall w \in D_W, \forall x" \neq_\Pi x'$, $x' \wedge y \wedge z' \wedge w >_\Pi x" \wedge y \wedge z' \wedge w$
  $\Rightarrow \forall y \in D_Y, \forall x" \neq_\Pi x'$, $max_w\{x' \wedge y \wedge z' \wedge w\} >_\Pi max_w\{x" \wedge y \wedge z' \wedge w\}$
  $\Rightarrow \forall y \in D_Y, \forall x" \neq_\Pi x'$ s.t $x' \wedge y \wedge z' >_\Pi x" \wedge y \wedge z'$
  $\Rightarrow \forall y \in D_Y, \mathbf{Acc}(x' \mid y \wedge z') = 1$.
  $\Rightarrow \mathbf{Acc}(x' \mid y' \wedge z') = 1$ (when Y takes y' as particular value)
  Hence contradiction.

- Weak union property for I_{BP}.
We want to prove that $I_{BP}(X, Y \mid Z \cup W) \Rightarrow I_{BP}(X, W \mid Z \cup Y)$.
Suppose that $I_{BP}(X, Y \mid Z \cup W)$ is true.
This means that $\forall x \in D_X, \forall y \in D_Y, \forall z \in D_Z, \forall w \in D_W$,
(a) $\mathbf{Acc}(x \mid y \wedge z \wedge w) = \mathbf{Acc}(x \mid z)$.
Moreover, we have shown that I_{BP} satisfies the decomposition property i.e.
$I_{BP}(X, Y \cup W \mid Z) \Rightarrow I_{BP}(X, Y \mid Z)$, namely, :
(b) if $\forall x \in D_X, \forall y \in D_Y, \forall z \in D_Z, \forall w \in D_W$, $\mathbf{Acc}(x \mid y \wedge z \wedge w) = \mathbf{Acc}(x \mid z)$
then $\forall x \in D_X, \forall y \in D_Y, \forall z \in D_Z, \forall w \in D_W$, $\mathbf{Acc}(x \mid y \wedge z) = \mathbf{Acc}(x \mid z)$.
Therefore from (a) and (b) we have $\forall x \in D_X, \forall y \in D_Y, \forall z \in D_Z, \forall w \in D_W$:
$\mathbf{Acc}(x \mid y \wedge z \wedge w) = \mathbf{Acc}(x \mid y \wedge z)$.
Hence $I_{BP}(X, W \mid Z \cup Y)$ is also true.

- Contraction property for I_{BP}.
We want to prove that $I_{BP}(X, W \mid Z \cup Y)$ and $I_{BP}(X, Y \mid Z) \Rightarrow I_{BP}(X, Y \cup W \mid Z)$.
Suppose that (i) $I_{BP}(X, W \mid Z \cup Y)$ and (ii) $I_{BP}(X, Y \mid Z)$ are true.
This means that $\forall x \in D_X, \forall y \in D_Y, \forall z \in D_Z, \forall w \in D_W$:
(a) $\mathbf{Acc}(x \mid y \wedge z \wedge w) = \mathbf{Acc}(x \mid y \wedge z)$ (from (i)) and
(b) $\mathbf{Acc}(x \mid y \wedge z) = \mathbf{Acc}(x \mid z)$ (from (ii))
(a) + (b) implies that $\forall x \in D_X, \forall y \in D_Y, \forall z \in D_Z, \forall w \in D_W$,
$\mathbf{Acc}(x \mid y \wedge z \wedge w) = \mathbf{Acc}(x \mid z)$.
Hence $I_{BP}(X, Y \mid Z \cup W)$ is also true.

- Intersection property for I_{BP}.
We want to prove that
$I_{BP}(X, Y \mid Z \cup W)$ and $I_{BP}(X, W \mid Z \cup Y) \Rightarrow I_{BP}(X, Y \cup W \mid Z)$.
Suppose that this relation is false.
Namely, we have $\forall x \in D_X, \forall y \in D_Y, \forall z \in D_Z, \forall w \in D_W$:
(i) $\mathbf{Acc}(x \mid y \wedge z \wedge w) = \mathbf{Acc}(x \mid z \wedge w)$ and (ii) $\mathbf{Acc}(x \mid y \wedge z \wedge w) = \mathbf{Acc}(x \mid y \wedge z)$
but $\exists x' \in D_X, y' \in D_Y, z' \in D_Z, w' \in D_W$, s.t.
(iii) $\mathbf{Acc}(x' \mid y' \wedge z' \wedge w') \neq \mathbf{Acc}(x' \mid z')$.

We distinguish three cases regarding the value of $\mathbf{Acc}(x' \mid y' \wedge z' \wedge w')$:

Case 1: $\mathbf{Acc}(x' \mid y' \wedge z' \wedge w') = 0$,
This implies from (i) that:
(iv) $\mathbf{Acc}(x' \mid z' \wedge w') = 0$.
Moreover, from (iii) $\mathbf{Acc}(x' \mid z')$ is either equal to -1 or 1, then:

- if $\mathbf{Acc}(x' \mid z') = -1$ then $\exists x" \neq_\Pi x' \in D_X$ s.t $x" \wedge z' >_\Pi x' \wedge z'$
  $\Rightarrow \exists x" \neq_\Pi x' \in D_X, \exists y" \in D_Y, \exists w" \in D_W$ s.t
  $x" \wedge z' >_\Pi x' \wedge y" \wedge z' \wedge w"$
  $\Rightarrow \exists y" \in D_Y, \exists z" \in D_Z$ s.t $\mathbf{Acc}(x' \mid y" \wedge z' \wedge w") = -1$
  $\Rightarrow \exists y" \in D_Y$ s.t $\mathbf{Acc}(x' \mid y" \wedge z') = -1$ (from (ii))

$\Rightarrow \exists y" \in D_Y$ s.t $\forall w \in D_W, \mathbf{Acc}(x' \mid y" \wedge z' \wedge w) = -1$ (from (ii))
$\Rightarrow \forall w \in D_W, \mathbf{Acc}(x' \mid z' \wedge w) = -1$ (from (i))
$\Rightarrow \mathbf{Acc}(x' \mid z' \wedge w') = -1$ (when W takes the particular instance w').
This contradicts (iv).

- if $\mathbf{Acc}(x' \mid z') = 1$
$\Rightarrow \forall x" \neq_\Pi x' \in D_X, x' \wedge z' >_\Pi x" \wedge z'$
$\Rightarrow \forall x" \neq_\Pi x' \in D_X, \exists y" \in D_Y, \exists z" \in D_Z$, s.t. $x' \wedge y" \wedge z' \wedge w" >_\Pi x" \wedge z'$
$\Rightarrow \forall x" \neq_\Pi x' \in D_X, \exists y" \in D_Y, \exists z" \in D_Z$ s.t. $x' \wedge y" \wedge z' \wedge w" >_\Pi x" \wedge y" \wedge z' \wedge w"$
$\Rightarrow \exists y" \in D_Y, \exists w" \in D_W$ s.t. $\mathbf{Acc}(x' \mid y" \wedge z' \wedge w") = 1$
$\Rightarrow \exists y" \in D_Y$ s.t. $\mathbf{Acc}(x' \mid y" \wedge z') = 1$ (from (ii))
$\Rightarrow \exists y" \in D_Y$ s.t. $\forall w \in D_W, \mathbf{Acc}(x' \mid y" \wedge z' \wedge w) = 1$ (from (ii))
$\Rightarrow \forall w \in D_W, \mathbf{Acc}(x' \mid z' \wedge w) = 1$ (from (i))
$\Rightarrow \mathbf{Acc}(x' \mid z' \wedge w') = 1$ (when W takes the particular instance w').
This contradicts (iv).

Case 2: $\mathbf{Acc}(x' \mid y' \wedge z' \wedge w') = 1$,
From (iii) $\mathbf{Acc}(x' \mid z')$ is either equal to -1 or 0.

- if $\mathbf{Acc}(x' \mid z') = -1$ then $\mathbf{Acc}(x' \mid z' \wedge w') = -1$
(by following same steps as in Case 1)
This contradicts (i) since $\mathbf{Acc}(x' \mid y' \wedge z' \wedge w') = 1$ (by assumption) and $\mathbf{Acc}(x' \mid z' \wedge w') = -1$.
- if $\mathbf{Acc}(x' \mid z') = 0$
$\Rightarrow \exists x" \neq_\Pi x' \in D_X$ s.t. $x" \wedge z' =_\Pi x' \wedge z'$ and $\forall x \in D_X, x" \wedge z' \geq_\Pi x \wedge z'$
$\Rightarrow \exists x" \neq_\Pi x' \in D_X$ s.t. $\forall x \in D_X, \forall y \in D_Y, \forall w \in D_W, x" \wedge z' \geq_\Pi x \wedge y \wedge z' \wedge w$
$\Rightarrow \exists x" \neq_\Pi x' \in D_X, \exists y" \in D_Y, \exists w" \in D_W$ s.t. $\forall x \in D_X$,
$x" \wedge y" \wedge z' \wedge w" \geq_\Pi x \wedge y" \wedge z' \wedge w"$
$\Rightarrow \exists x" \neq_\Pi x' \in D_X, \exists y" \in D_Y, \exists w" \in D_W$, s.t. $\mathbf{Acc}(x" \mid y" \wedge z' \wedge w") \geq 0$
$\Rightarrow \exists x" \neq_\Pi x' \in D_X, \exists y" \in D_Y$, s.t. $\mathbf{Acc}(x" \mid y" \wedge z') \geq 0$ (from (ii))
$\Rightarrow \exists x" \neq_\Pi x' \in D_X, \exists y" \in D_Y$, s.t. $\forall w \in D_W, \mathbf{Acc}(x" \mid y" \wedge z' \wedge w) \geq 0$
(from (ii))
$\Rightarrow$ (a) $\forall w \in D_W, \mathbf{Acc}(x" \mid z' \wedge w) \geq 0$ (from (i))
Moreover,
$\mathbf{Acc}(x' \mid y' \wedge z' \wedge w') = 1$
$\Rightarrow \mathbf{Acc}(x' \mid y' \wedge z') = 1$ (from (ii))
$\Rightarrow \forall w \in D_W, \mathbf{Acc}(x' \mid y' \wedge z' \wedge w) = 1$ (from (ii))
$\Rightarrow \forall w \in D_W, \mathbf{Acc}(x' \mid z' \wedge w) = 1$ (from (i))
$\Rightarrow \forall x \neq_\Pi x' \in D_X, \forall w \in D_W, \mathbf{Acc}(x \mid z' \wedge w) = -1$ (by definition of $\mathbf{Acc}$)
Hence this contradicts (a) when X takes the particular instance $x" \neq_\Pi x'$.

Case 3: $\mathbf{Acc}(x' \mid y' \wedge z' \wedge w') = -1$,
From (iii) we have $\mathbf{Acc}(x' \mid z')$ is either equal to 0 or 1.

- if $\mathbf{Acc}(x' \mid z') = 0$
$\Rightarrow \forall x \in D_X, x' \wedge z' \geq_\Pi x \wedge z'$

$\Rightarrow \forall x \in D_X, \forall y \in D_Y, \forall w \in D_W, x' \wedge z' \geq_\Pi x \wedge y \wedge z' \wedge w$
$\Rightarrow \exists y" \in D_Y, \exists w" \in D_W$ s.t. $\forall x \in D_X, x' \wedge y" \wedge z' \wedge w" \geq_\Pi x \wedge y" \wedge z' \wedge w"$
$\Rightarrow \exists y" \in D_Y, \exists w" \in D_W$, s.t. $\mathbf{Acc}(x' \mid y" \wedge z' \wedge w") \geq 0$
$\Rightarrow \exists y" \in D_Y$, s.t. $\mathbf{Acc}(x' \mid z' \wedge y") \geq 0$ (from (ii))
$\Rightarrow \exists y" \in D_Y$, s.t. $\forall w \in D_W, \mathbf{Acc}(x' \mid y" \wedge z' \wedge w) \geq 0$
(from (ii))
$\Rightarrow$ (b) $\forall w \in D_W, \mathbf{Acc}(x' \mid z' \wedge w) \geq 0$ (from (i))
Moreover,
$\mathbf{Acc}(x' \mid y' \wedge z' \wedge w') = -1$
$\Rightarrow \mathbf{Acc}(x' \mid y' \wedge z') = -1$ (from (ii))
$\Rightarrow \forall w \in D_W, \mathbf{Acc}(x' \mid y' \wedge z' \wedge w) = -1$ (from (ii))
$\Rightarrow \forall w \in D_W, \mathbf{Acc}(x' \mid z' \wedge w) = -1$ (from (i))
Hence contradiction with (b).

- if $\mathbf{Acc}(x' \mid z') = 1$ then $\mathbf{Acc}(x' \mid z' \wedge w') = 1$ (by following same steps as in Case 1)
  Hence, this contradicts (i) since $\mathbf{Acc}(x' \mid y' \wedge z' \wedge w') = -1$ (by assumption) and $\mathbf{Acc}(x' \mid z' \wedge w') = 1$.

- Reverse-Decomposition property for I_{BP}.
We want to prove that $I_{BP}(X \cup Y, W \mid Z) \Rightarrow I_{BP}(Y, W \mid Z)$ and $I_{BP}(X, W \mid Z)$.
We only prove that $I_{BP}(X \cup Y, W \mid Z) \Rightarrow I_{BP}(X, W \mid Z)$
(the proof of $I_{BP}(X \cup Y, W \mid Z) \Rightarrow I_{BP}(Y, W \mid Z)$ is analogous).
This means that we want to prove that:
if (i) $\forall x \in D_X, \forall y \in D_Y, \forall z \in D_Z, \forall w \in D_W, \mathbf{Acc}(x \wedge y \mid z \wedge w) = \mathbf{Acc}(x \wedge y \mid z)$
then (ii) $\forall x \in D_X, y \in D_Y, z \in D_Z, w \in D_W, \mathbf{Acc}(x \mid z \wedge w) = \mathbf{Acc}(x \mid z)$.
Assume that (i) is true, and let us show that (ii) is also true.
Let us analyze the possible values of $\mathbf{Acc}(x \mid z)$:

Case 1: $\mathbf{Acc}(x \mid z) = 0$
$\Rightarrow \exists x' \neq_\Pi x \in D_X$ s.t. $x \wedge z =_\Pi x' \wedge z$ and
$\forall x" \in D_X, x \wedge z \geq_\Pi x" \wedge z$ and $x' \wedge z \geq_\Pi x" \wedge z$.
$\Rightarrow \exists x' \neq_\Pi x \in D_X$, s.t. $max_{y"}\{x \wedge y" \wedge z\} =_\Pi max_{y"}\{x' \wedge y" \wedge z\}$ and
$\forall x" \in D_X, max_{y"}\{x \wedge y" \wedge z\} =_\Pi max_{y"}\{x' \wedge y" \wedge z\} >_\Pi max_{y"}\{x" \wedge y" \wedge z\}$
$\Rightarrow \exists x' \neq_\Pi x \in D_X, \exists y, y' \in D_Y$ s.t. $\mathbf{Acc}(x \wedge y \mid z) = \mathbf{Acc}(x' \wedge y' \mid z) = 0$
(It is enough to take y and y' such that $x \wedge y \wedge z = max_{y"}\{x \wedge y" \wedge z\}$ and
$x' \wedge y' \wedge z = max_{y"}\{x' \wedge y" \wedge z\}$)
$\Rightarrow \exists x' \neq_\Pi x \in D_X, \exists y, y' \in D_Y$ s.t.
$\forall w \in D_W, \mathbf{Acc}(x \wedge y \mid z \wedge w) = \mathbf{Acc}(x' \wedge y' \mid z \wedge w) = 0$ (from (i))
$\Rightarrow \exists x' \neq_\Pi x \in D_X, \exists y, y' \in D_Y$ s.t.
$\forall x" \in D_X, \forall y" \in D_Y, \forall w \in D_W, x \wedge y \wedge z \wedge w =_\Pi x' \wedge y' \wedge z \wedge w \geq_\Pi x" \wedge y" \wedge z \wedge w$
$\Rightarrow \exists x' \neq x \in D_X, \exists y, y' \in D_Y$ s.t.
$\forall x" \in D_X, \forall w \in D_W, x \wedge y \wedge z \wedge w =_\Pi x' \wedge y' \wedge z \wedge w \geq_\Pi max_{y"}\{x" \wedge y" \wedge z \wedge w\}$
$\Rightarrow \exists x' \neq_\Pi x \in D_X, \exists y, y' \in D_Y$ s.t.
(a) $\forall x" \in D_X, \forall w \in D_W$, $x \wedge y \wedge z \wedge w =_\Pi x' \wedge y' \wedge z \wedge w \geq_\Pi x" \wedge z \wedge w$

$\Rightarrow \exists x' \neq_\Pi x \in D_X, \exists y, y' \in D_Y$ s.t. $\forall x" \in D_X, \forall w \in D_W$,
$x \wedge y \wedge z \wedge w \geq_\Pi x \wedge z \wedge w$, (from (a) when $x" =_\Pi x$)
$x' \wedge y' \wedge z \wedge w \geq_\Pi x' \wedge z \wedge w$ (from (a) when $x" =_\Pi x'$), and
$x \wedge y \wedge z \wedge w =_\Pi x' \wedge y' \wedge z \wedge w \geq_\Pi x" \wedge z \wedge w$ (from (a))
$\Rightarrow \exists x' \neq_\Pi x \in D_X, \exists y, y' \in D_Y$ s.t. $\forall x" \in D_X, \forall w \in D_W$,
$x \wedge y \wedge z \wedge w =_\Pi x \wedge z \wedge w$,
$x' \wedge y' \wedge z \wedge w =_\Pi x' \wedge z \wedge w$ and
$x \wedge y \wedge z \wedge w =_\Pi x' \wedge y' \wedge z \wedge w \geq_\Pi x" \wedge z \wedge w$
$\Rightarrow \exists x' \neq_\Pi x \in D_X$ s.t. $\forall x" \in D_X, \forall w \in D_W$,
$x \wedge z \wedge w =_\Pi x' \wedge z \wedge w \geq_\Pi x" \wedge z \wedge w$
$\Rightarrow \forall w \in D_W, \mathbf{Acc}(x \mid z \wedge w) = 0$.

Case 2: $\mathbf{Acc}(x \mid z) = 1$
$\Rightarrow \forall x' \neq_\Pi x \in D_X,\ x \wedge z >_\Pi x' \wedge z$
$\Rightarrow \forall x' \neq_\Pi x \in D_X, \exists w' \in D_W$ s.t. $x \wedge z \wedge w' >_\Pi x' \wedge z \wedge w'$
$\Rightarrow \forall x' \neq_\Pi x \in D_X, \exists w' \in D_W$ s.t. $\forall y \in D_Y$ we have $x \wedge z \wedge w' >_\Pi x' \wedge y \wedge z \wedge w'$
$\Rightarrow \forall x' \neq_\Pi x \in D_X, \exists w' \in D_W$ s.t. $\forall y \in D_Y$ we have $\mathbf{Acc}(x' \wedge y \mid z \wedge w') = -1$
$\Rightarrow \forall x' \neq_\Pi x \in D_X, \forall y \in D_Y,\ \mathbf{Acc}(x' \wedge y \mid z) = -1$ (from (i))
$\Rightarrow \forall x' \neq_\Pi x \in D_X, \forall y \in D_Y, \forall w \in D_W,\ \mathbf{Acc}(x' \wedge y \mid z \wedge w) = -1$ (from (i))
$\Rightarrow \forall x' \neq_\Pi x \in D_X, \forall y \in D_Y, \forall w \in D_W,\ x' \wedge y \wedge z \wedge w <_\Pi x \wedge y \wedge z \wedge w$
$\Rightarrow \forall x' \neq_\Pi x \in D_X, \forall w \in D_W, max_{y'}\{x' \wedge y \wedge z \wedge w\} <_\Pi max_{y'}\{x \wedge y \wedge z \wedge w\}$
$\Rightarrow \forall x' \neq_\Pi x \in D_X, \forall w \in D_W, x' \wedge z \wedge w <_\Pi x \wedge z \wedge w$
$\Rightarrow \forall x' \neq_\Pi x \in D_X, \forall w \in D_W, \mathbf{Acc}(x' \mid z \wedge w) = -1$
$\Rightarrow \forall w \in D_W,\ \mathbf{Acc}(x \mid z \wedge w) = 1$.

Case 3: $\mathbf{Acc}(x \mid z) = -1$
$\Rightarrow \exists x' \neq_\Pi x \in D_X,\ x' \wedge z >_\Pi x \wedge z$
$\Rightarrow \exists x' \neq_\Pi x \in D_X, \exists w' \in D_W$ s.t. $x' \wedge z \wedge w' >_\Pi x \wedge z \wedge w'$
$\Rightarrow \exists x' \neq_\Pi x \in D_X, \exists y' \in D_Y, \exists w' \in D_W$ s.t. $x' \wedge y' \wedge z \wedge w' >_\Pi x \wedge z \wedge w'$
$\Rightarrow \exists x' \neq_\Pi x \in D_X, \exists y' \in D_Y, \exists w' \in D_W$ s.t. $\forall y \in D_Y$ we have
$x' \wedge y' \wedge z \wedge w' >_\Pi x \wedge y \wedge z \wedge w'$
$\Rightarrow \forall y \in D_Y, \exists w' \in D_W$ s.t. $\mathbf{Acc}(x \wedge y \mid z \wedge w') = -1$
$\Rightarrow \forall y \in D_Y, \mathbf{Acc}(x \wedge y \mid z) = -1$ (from (i))
$\Rightarrow \forall y \in D_Y, \forall w \in D_W, \mathbf{Acc}(x \wedge y \mid z \wedge w) = -1$ (from (i))
$\Rightarrow \forall w \in D_W, \mathbf{Acc}(x \mid z \wedge w) = -1$.

Proof of Proposition 2
- Decomposition property for I_{PO}.
We want to prove that $I_{PO}(X, Y \cup W \mid Z) \Rightarrow I_{PO}(X, Y \mid Z)$ and $I_{PO}(X, W \mid Z)$
We only prove that $I_{PO}(X, Y \cup W \mid Z) \Rightarrow I_{PO}(X, Y \mid Z)$
(the proof of $I_{PO}(X, Y \cup W \mid Z) \Rightarrow I_{PO}(X, W \mid Z)$ is analogous).

Thus, we need to prove that:
if (i) $\forall y \in D_Y, \forall z \in D_Z, \forall w \in D_W$:
$\forall x_i, x_j \in D_X, x_i \wedge z >_\Pi x_j \wedge z$ iff $x_i \wedge y \wedge z \wedge w >_\Pi x_j \wedge y \wedge z \wedge w$
then (ii) $\forall y \in D_Y, \forall z \in D_Z$:
$\forall x_i, x_j \in D_X, x_i \wedge z >_\Pi x_j \wedge z$ iff $x_i \wedge y \wedge z >_\Pi x_j \wedge y \wedge z$.
Let us assume that (i) is true, and let us show that (ii) is also true.

Let $z \in D_Z$ and $x_i, x_j \in D_X$, and assume that $x_i \wedge z >_\Pi x_j \wedge z$
(resp. $x_i \wedge z =_\Pi x_j \wedge z$)
$\Rightarrow \forall y \in D_Y, \forall w \in D_W$, $x_i \wedge y \wedge z \wedge w >_\Pi x_j \wedge y \wedge z \wedge w$
(resp. $x_i \wedge y \wedge z \wedge w =_\Pi x_j \wedge y \wedge z \wedge w$) (from (i))
$\Rightarrow \forall y \in D_Y, max_w\{x_i \wedge y \wedge z \wedge w\} >_\Pi max_w\{x_j \wedge y \wedge z \wedge w\}$
(resp. $max_w\{x_i \wedge y \wedge z \wedge w\} =_\Pi max_w\{x_j \wedge y \wedge z \wedge w\}$)
$\Rightarrow \forall y \in D_Y, x_i \wedge y \wedge z >_\Pi x_j \wedge y \wedge z$ (resp. $x_i \wedge y \wedge z =_\Pi x_j \wedge y \wedge z$).

- Weak union property for I_{PO}.
We want to prove that $I_{PO}(X, Y \cup W \mid Z) \Rightarrow I_{PO}(X, W \mid Z \cup Y)$.
Thus, we need to prove that:
if (i) $\forall y \in D_Y, \forall z \in D_Z, \forall w \in D_W$:
$\forall x_i, x_j \in D_X, x_i \wedge z >_\Pi x_j \wedge z$ iff $x_i \wedge y \wedge z \wedge w >_\Pi x_j \wedge y \wedge z \wedge w$
then (ii) $\forall y \in D_Y, \forall z \in D_Z, \forall w \in D_W$:
$\forall x_i, x_j \in D_X, x_i \wedge y \wedge z >_\Pi x_j \wedge y \wedge z$ iff $x_i \wedge y \wedge z \wedge w >_\Pi x_j \wedge y \wedge z \wedge w$.
Let us assume that (i) is true, and let us show that (ii) is also true.

Let $x_i, x_j \in D_X$, $y' \in D_Y$ and $w' \in D_W$ and $z \in D_Z$,
Assume that $x_i \wedge y' \wedge z \wedge w' >_\Pi x_j \wedge y' \wedge z \wedge w'$ (resp. $x_i \wedge y' \wedge z \wedge w' =_\Pi x_j \wedge y' \wedge z \wedge w'$)
$\Rightarrow \forall z \in D_Z, x_i \wedge z >_\Pi x_j \wedge z$ (resp. $x_i \wedge z =_\Pi x_j \wedge z$) (from (i))
$\Rightarrow \forall y \in D_Y, \forall w \in D_W, x_i \wedge y \wedge z \wedge w >_\Pi x_j \wedge y \wedge z \wedge w$
(resp. $x_i \wedge y \wedge z \wedge w =_\Pi x_j \wedge y \wedge z \wedge w$) (from (i))
$\Rightarrow \forall y \in D_Y, max_w\{x_i \wedge y \wedge z \wedge w\} >_\Pi max_w\{x_j \wedge y \wedge z \wedge w\}$
(resp. $max_w\{x_i \wedge y \wedge z \wedge w\} =_\Pi max_w\{x_j \wedge y \wedge z \wedge w\}$)
$\Rightarrow \forall y \in D_Y, x_i \wedge y \wedge z >_\Pi x_j \wedge y \wedge z$ (resp. $x_i \wedge y \wedge z =_\Pi x_j \wedge y \wedge z$)
$\Rightarrow x_i \wedge y' \wedge z >_\Pi x_j \wedge y' \wedge z$ (resp. $x_i \wedge y' \wedge z =_\Pi x_j \wedge y' \wedge z$) (when Y takes the particular instance y')

- Contraction property for I_{PO}.
We want to prove that $I_{PO}(X, W \mid Z \cup Y)$ and $I_{PO}(X, Y \mid Z) \Rightarrow I_{PO}(X, Y \cup W \mid Z)$.
Thus, we need to prove that:
if (i) $\forall y \in D_Y, \forall z \in D_Z, \forall w \in D_W$:
$\forall x_i, x_j \in D_X, x_i \wedge y \wedge z >_\Pi x_j \wedge y \wedge z$ iff $x_i \wedge y \wedge z \wedge w >_\Pi x_j \wedge y \wedge z \wedge w$ and
(ii) $\forall y \in D_Y, \forall z \in D_Z$:
$\forall x_i, x_j \in D_X, x_i \wedge z >_\Pi x_j \wedge z$ iff $x_i \wedge y \wedge z >_\Pi x_j \wedge y \wedge z$
then (iii) $\forall y \in D_Y, \forall z \in D_Z, \forall w \in D_W$:

$\forall x_i, x_j \in D_X, x_i \wedge z >_\Pi x_j \wedge z$ iff $x_i \wedge y \wedge z \wedge w >_\Pi x_j \wedge y \wedge z \wedge w$.
Let us assume that (i) and (ii) are true, and let us show that (iii) is also true.

Let $x_i, x_j \in D_X$ and $\forall z \in D_Z$.
Assume that $x_i \wedge z >_\Pi x_j \wedge z$
(resp. $x_i \wedge z =_\Pi x_j \wedge z$)
$\Rightarrow \forall y \in D_Y, x_i \wedge y \wedge z >_\Pi x_j \wedge y \wedge z$
(resp. $x_i \wedge y \wedge z =_\Pi x_j \wedge y \wedge z$) (from (ii))
$\Rightarrow \forall y \in D_Y, \forall w \in D_W, x_i \wedge y \wedge z \wedge w >_\Pi x_j \wedge y \wedge z \wedge w$
(resp. $x_i \wedge y \wedge z \wedge w =_\Pi x_j \wedge y \wedge z \wedge w$) (from (i)).

- Intersection property for I_{PO}.
We want to prove that
$I_{PO}(X, Y \mid Z \cup W)$ and $I_{PO}(X, W \mid Z \cup Y) \Rightarrow I_{PO}(X, Y \cup W \mid Z)$
Thus we need to prove that:
if (i) $\forall y \in D_Y, \forall z \in D_Z, \forall w \in D_W$:
$\forall x_i, x_j \in D_X, x_i \wedge y \wedge z \wedge w >_\Pi x_j \wedge y \wedge z \wedge w$ iff $x_i \wedge z \wedge w >_\Pi x_j \wedge z \wedge w$ and
(ii) $\forall z \in D_Z, \forall y \in D_Y, \forall w \in D_W$:
$\forall x_i, x_j \in D_X, x_i \wedge y \wedge z \wedge w >_\Pi x_j \wedge y \wedge z \wedge w$ iff $x_i \wedge y \wedge z >_\Pi x_j \wedge y \wedge z$
then (iii) $\forall z \in D_Z, \forall y \in D_Y, \forall w \in D_W$:
$\forall x_i, x_j \in D_X, x_i \wedge y \wedge z \wedge w >_\Pi x_j \wedge y \wedge z \wedge w$ iff $x_i \wedge z >_\Pi x_j \wedge z$
Let us assume that (i) and (ii) are true, and that (iii) is false.

This means that: $\exists x', x" \in D_X, \exists y' \in D_Y, \exists z' \in D_Z, \exists w' \in D_W$ s.t.
$x' \wedge z' >_\Pi x" \wedge z'$ (resp. $x' \wedge z' =_\Pi x" \wedge z'$)
but
$x" \wedge y' \wedge z' \wedge w' \geq_\Pi x' \wedge y' \wedge z' \wedge w'$ (resp. $x' \wedge y' \wedge z' \wedge w' \neq_\Pi x" \wedge y' \wedge z' \wedge w'$)
$\Rightarrow x" \wedge y' \wedge z' \geq_\Pi x' \wedge y' \wedge z'$ (resp. $x' \wedge y' \wedge z' \neq_\Pi x" \wedge y' \wedge z'$) (from (ii))
$\Rightarrow \forall w \in D_W, x" \wedge y' \wedge z' \wedge w \geq_\Pi x' \wedge y' \wedge z' \wedge w$ (resp. $x' \wedge y' \wedge z' \wedge w \neq_\Pi x" \wedge y' \wedge z' \wedge w$)
(from (ii))
$\Rightarrow \forall w \in D_W, x" \wedge z' \wedge w \geq_\Pi x' \wedge z' \wedge w$ (resp. $x' \wedge z' \wedge w \neq_\Pi x" \wedge z' \wedge w$) (from (i))
$\Rightarrow max_w\{x" \wedge z' \wedge w\} \geq_\Pi max_w\{x' \wedge z' \wedge w\}$
(resp. $max_w\{x' \wedge z' \wedge w\} \neq_\Pi max_w\{x" \wedge z' \wedge w\}$)
$\Rightarrow x" \wedge z' >_\Pi x' \wedge z'$ (resp. $x' \wedge z' \neq_\Pi x" \wedge z'$)
Hence contradiction.

- Reverse-Decomposition property for I_{PO}.
We want to prove that $I_{PO}(X \cup Y, W \mid Z) \Rightarrow I_{PO}(Y, W \mid Z)$ and $I_{PO}(X, W \mid Z)$.
We only prove that $I_{PO}(X \cup Y, W \mid Z) \Rightarrow I_{PO}(Y, W \mid Z)$
(the proof of $I_{PO}(X \cup Y, W \mid Z) \Rightarrow I_{PO}(X, W \mid Z)$ is analogous).
Thus, we need to prove that:
if (i) $\forall z \in D_Z, \forall w \in D_W$:
$\forall x_k, x_l \in D_X, \forall y_m, y_n \in D_Y, x_k \wedge y_m \wedge z >_\Pi x_l \wedge y_n \wedge z$ iff

$x_k \wedge y_m \wedge z \wedge w >_\Pi x_l \wedge y_n \wedge z \wedge w$
then (ii) $\forall z \in D_Z, \forall w \in D_W$:
$\forall y_k, y_l \in D_Y, y_k \wedge z >_\Pi y_l \wedge z$ iff $y_k \wedge z \wedge w >_\Pi y_l \wedge z \wedge w$.
Let us assume that (i) is true and let us show that (ii) is also true.

Let $z \in D_Z$ and let $y_k, y_l \in D_Y$ s.t. $y_k \wedge z >_\Pi y_l \wedge z$ (resp. $y_k \wedge z =_\Pi y_l \wedge z$).
Let x_i be one of the instances of X which maximizes $y_k \wedge z$ and x_j one of the instances of X which maximizes $y_l \wedge z$.
Namely $x_i \wedge y_k \wedge z =_\Pi max_x\{x \wedge y_k \wedge z\}$ and $x_j \wedge y_l \wedge z =_\Pi max_x\{x \wedge y_l \wedge z\}$
Then:
$x_i \wedge y_k \wedge z >_\Pi x_j \wedge y_l \wedge z$ (resp. $x_i \wedge y_k \wedge z =_\Pi x_j \wedge y_l \wedge z$)
$\Rightarrow \forall w \in D_W, x_i \wedge y_k \wedge z \wedge w >_\Pi x_j \wedge y_l \wedge z \wedge w$
(resp. $\forall w \in D_W, x_i \wedge y_k \wedge z \wedge w =_\Pi x_j \wedge y_l \wedge z \wedge w$) (from (i))
$\Rightarrow \forall w \in D_W, max_x\{x \wedge y_k \wedge z \wedge w\} =_\Pi x_i \wedge y_k \wedge z \wedge w >_\Pi max_x\{x \wedge y_l \wedge z \wedge w\} = x_j \wedge y_l \wedge z \wedge w$ (resp. $\forall w \in D_W, max_x\{x \wedge y_k \wedge z \wedge w\} =_\Pi x_i \wedge y_k \wedge z \wedge w =_\Pi max_x\{x \wedge y_l \wedge z \wedge w\} =_\Pi x_j \wedge y_l \wedge z \wedge w$)
$\Rightarrow \forall w \in D_W, y_k \wedge z \wedge w >_\Pi y_l \wedge z \wedge w$ (resp. $\forall w \in D_W, y_k \wedge z \wedge w =_\Pi y_l \wedge z \wedge w$).

- Reverse-Weak union property for I_{PO}.
We want to prove that $I_{PO}(X \cup Y, W \mid Z) \Rightarrow I_{PO}(X, W \mid Y \cup Z)$.
Thus, we need to prove that:
if (i) $\forall z \in D_Z, \forall w \in D_W : \forall x_k, x_l \in D_X, \forall y_m, y_n \in D_Y$,
$x_k \wedge y_m \wedge z >_\Pi x_l \wedge y_n \wedge z$ iff $x_k \wedge y_m \wedge z \wedge w >_\Pi x_l \wedge y_n \wedge z \wedge w$
then (ii) $\forall y \in D_Y, \forall z \in D_Z, \forall w \in D_W : \forall x_i, x_j \in D_X$,
$x_i \wedge y \wedge z >_\Pi x_j \wedge y \wedge z$ iff $x_i \wedge y \wedge z \wedge w >_\Pi x_j \wedge y \wedge z \wedge w$

The proof is immediate.
Indeed, $\exists x_i, x_j \in D_X$, $\exists y' \in D_Y$ and $\exists z' \in D_Z$ s.t. $x_i \wedge y' \wedge z' >_\Pi x_j \wedge y' \wedge z'$ (resp. $x_i \wedge y' \wedge z' =_\Pi x_j \wedge y' \wedge z'$).
This means that $\forall w \in D_W, x_i \wedge y' \wedge z' \wedge w >_\Pi x_j \wedge y' \wedge z' \wedge w$ (resp. $x_i \wedge y' \wedge z' \wedge w =_\Pi x_j \wedge y' \wedge z' \wedge w$) (from (i)).

Proof of Proposition 5
- Decomposition property for I_{PT}.
We want to prove that
$I_{PT}(X, Y \cup W \mid Z) \Rightarrow I_{PT}(X, Y \mid Z)$ and $I_{PT}(X, W \mid Z)$.
We only prove that $I_{PT}(X, Y \cup W \mid Z) \Rightarrow I_{PT}(X, Y \mid Z)$
(the proof of $I_{PT}(X, Y \cup W \mid Z) \Rightarrow I_{PT}(X, W \mid Z)$ is analogous).
Suppose that this relation is false. Namely, we have:
(i) $\forall x \in D_X, \forall y \in D_Y, \forall w \in D_W$,
$\mathbf{Acc}(x \wedge y \wedge w \mid z) = min(\mathbf{Acc}(x \mid z), \mathbf{Acc}(y \wedge w \mid z))$
but (ii) $\exists x' \in D_X, \exists y' \in D_Y, \exists z' \in D_Z$ s.t.
$\mathbf{Acc}(x' \wedge y' \mid z') \neq min(\mathbf{Acc}(x' \mid z'), \mathbf{Acc}(y' \mid z'))$.

Using Lemma 1, the unique case where this inequality holds is when:
(a) $\mathbf{Acc}(x' \wedge y' \mid z') = -1$, (b) $\mathbf{Acc}(x' \mid z') = 0$ and (c) $\mathbf{Acc}(y' \mid z') = 0$.
The equality (c) implies that $\exists y" \neq_{\Pi} y' \in D_Y$ s.t. $y" \wedge z' =_{\Pi} y' \wedge z'$ and y' and $y"$ are accepted instances of Y in the context of z'.
Namely, $\forall y \in D_Y, y" \wedge z' \geq_{\Pi} y \wedge z'$ and $y' \wedge z' \geq_{\Pi} y \wedge z'$.
By definition we have:
$y' \wedge z' =_{\Pi} max_w\{y' \wedge z' \wedge w\}$
$y" \wedge z' =_{\Pi} max_w\{y" \wedge z' \wedge w\}$
Let $w_i, w_j \in D_W$ s.t. $max_w\{y' \wedge z' \wedge w\} =_{\Pi} y' \wedge z' \wedge w_i$ and $max_w\{y" \wedge z' \wedge w\} =_{\Pi} y" \wedge z' \wedge w_j$
$\Rightarrow y' \wedge z' \wedge w_i =_{\Pi} y" \wedge z' \wedge w_j$ (since $y' \wedge z' =_{\Pi} y" \wedge z'$)
$\Rightarrow \mathbf{Acc}(y' \wedge w_i \mid z') = 0$ (since $y' \wedge z' \wedge w_i =_{\Pi} max_w\{y' \wedge z' \wedge w\}$ and $y" \wedge z' \wedge w_j =_{\Pi} max_w\{y" \wedge z' \wedge w\}$ and $y" \neq_{\Pi} y'$)
Moreover, (a) implies that $\exists x_k \in D_X, \exists y_k \in D_Y$ s.t. $x_k \wedge y_k \wedge z' >_{\Pi} x' \wedge y' \wedge z'$
$\Rightarrow \exists x_k \in D_X, \exists y_k \in D_Y, \exists w_k \in D_W$ s.t. $x_k \wedge y_k \wedge z' \wedge w_k >_{\Pi} x' \wedge y' \wedge z'$
$\Rightarrow \exists x_k \in D_X, \exists y_k \in D_Y, \exists w_k \in D_W$ s.t. $\forall w \in D_W, x_k \wedge y_k \wedge z' \wedge w_k >_{\Pi} x' \wedge y' \wedge z' \wedge w$
$\Rightarrow \exists x_k \in D_X, \exists y_k \in D_Y, \exists w_k \in D_W$ s.t. $x_k \wedge y_k \wedge z' \wedge w_k >_{\Pi} x' \wedge y' \wedge z' \wedge w_i$
(when W takes w_i as particular value)
$\Rightarrow \mathbf{Acc}(x' \wedge y' \wedge w_i \mid z') = -1$
$\Rightarrow \mathbf{Acc}(x' \wedge y' \wedge w_i \mid z') \neq min(\mathbf{Acc}(x' \mid z'), \mathbf{Acc}(y' \wedge w_i \mid z')) = 0$
Hence contradiction with (i).

- Contraction property for I_{PT}.
We want to prove that $I_{PT}(X, W \mid Z \cup Y)$ and $I_{PT}(X, Y \mid Z) \Rightarrow I_{PT}(X, Y \cup W \mid Z)$
Suppose that this relation is false. This means that we have:
(i) $\forall x \in D_X, \forall y \in D_Y, \forall z \in D_Z, \forall w \in D_W$,
$\mathbf{Acc}(x \wedge w \mid y \wedge z) = min(\mathbf{Acc}(x \mid y \wedge z), \mathbf{Acc}(w \mid y \wedge z))$
(ii) $\forall x \in D_X, \forall y \in D_Y, \forall z \in D_Z, \mathbf{Acc}(x \wedge y \mid z) = min(\mathbf{Acc}(x \mid z), \mathbf{Acc}(y \mid z))$
but
(iii) $\exists x' \in D_X, \exists y' \in D_Y, \exists z' \in D_Z, \exists w' \in D_W$ s.t.
$\mathbf{Acc}(x' \wedge y' \wedge w' \mid z') \neq min(\mathbf{Acc}(x' \mid z'), \mathbf{Acc}(y' \wedge w' \mid z'))$.

Using Lemma 1, the unique case where this inequality holds is when
(a) $\mathbf{Acc}(x' \wedge y' \wedge w' \mid z') = -1$, (b) $\mathbf{Acc}(x' \mid z') = 0$ and (c) $\mathbf{Acc}(y' \wedge w' \mid z') = 0$.
Let us analyze the three possible values for $\mathbf{Acc}(x' \wedge y' \mid z')$:

- **Case 1**: $\mathbf{Acc}(x' \wedge y' \mid z') = -1$
  Using (ii) we have $\mathbf{Acc}(x' \mid z') = -1$.
  Hence, this contradicts (b).
- **Case 2**: $\mathbf{Acc}(x' \wedge y' \mid z') = 1$
  $\Rightarrow \mathbf{Acc}(y' \mid z') = 1$ and $\mathbf{Acc}(x' \mid z') = 1$ (from Lemma 2)
  Hence this contradicts (b).
- **Case 3**: $\mathbf{Acc}(x' \wedge y' \mid z') = 0$

From (a) we have:
$\exists x" \in D_X, \exists y" \in D_Y, \exists w" \in D_W$ s.t $x" \wedge y" \wedge z' \wedge w" >_\Pi x' \wedge y' \wedge z' \wedge w'$
From (c) we have:
$\forall y \in D_Y, \forall w \in D_W, y' \wedge z' \wedge w' \geq_\Pi y \wedge z' \wedge w$
$\Rightarrow y' \wedge z' \wedge w' \geq_\Pi y" \wedge z' \wedge w"$
(when Y and W takes $y"$ and $w"$ as particular values)
$\Rightarrow \exists x \in D_X$ s.t. $x \wedge y' \wedge z' \wedge w' \geq_\Pi y" \wedge z' \wedge w" \geq_\Pi x" \wedge y" \wedge z' \wedge w" >_\Pi x' \wedge y' \wedge z' \wedge w'$
(since by definition $y" \wedge z' \wedge w" \geq_\Pi x" \wedge y" \wedge z' \wedge w"$)
$\Rightarrow \exists x \in D_X$ s.t. $x \wedge y' \wedge z' \wedge w' >_\Pi x' \wedge y' \wedge z' \wedge w'$
$\Rightarrow \mathbf{Acc}(x' \wedge w' \mid y' \wedge z') = -1$
Moreover $\mathbf{Acc}(x' \wedge y' \mid z') = 0$ implies that $\mathbf{Acc}(x' \mid y' \wedge z') \geq 0$ and
(c) implies that $\mathbf{Acc}(w' \mid y' \wedge z') \geq 0$.
Hence, contradiction with (i).

Proof of Proposition 7
- Decomposition property for $I_{leximax}$.
We want to prove that
$I_{leximax}(X, Y \cup W \mid Z) \Rightarrow I_{leximax}(X, Y \mid Z)$ and $I_{leximax}(X, W \mid Z)$.
We only prove that if $I_{leximax}(X, Y \cup W \mid Z)$ is true then $I_{leximax}(X, Y \mid Z)$ is true (the proof of $I_{leximax}(X, Y \cup W \mid Z) \Rightarrow I_{leximax}(X, W \mid Z)$ is analogous).

Suppose that
(i) $I_{leximax}(X, Y \cup W \mid Z)$ is true
but not $I_{leximax}(X, Y \mid Z)$.
Let us consider the two cases where $I_{leximax}(X, Y \mid Z)$ is falsified:

Case 1: $\exists x, x' \in D_X, \exists y, y' \in D_Y, \exists z' \in D_Z$ s.t.
(a) $x \wedge y \wedge z' >_\Pi x' \wedge y' \wedge z'$ but
(i1) $max(x \wedge z', y \wedge z') <_\Pi max(x' \wedge z', y' \wedge z')$ or
(i2) $max(x \wedge z', y \wedge z') =_\Pi max(x' \wedge z', y' \wedge z')$ and
$min(x \wedge z', y \wedge z') \leq_\Pi min(x' \wedge z', y' \wedge z')$

By definition we have $x \wedge y \wedge z' =_\Pi max_w\{x \wedge y \wedge z' \wedge w\}$ and
$x' \wedge y' \wedge z' =_\Pi max_w\{x' \wedge y' \wedge z' \wedge w\}$
Let w_i be one of the instances of W which maximizes $x \wedge y \wedge z'$ and w_j be one of the instances of W which maximizes $x' \wedge y' \wedge z'$. Namely,
$x \wedge y \wedge z' =_\Pi x \wedge y \wedge z' \wedge w_i$ and $x' \wedge y' \wedge z' =_\Pi x' \wedge y' \wedge z' \wedge w_j$

From (a) we have $x \wedge y \wedge z' \wedge w_i >_\Pi x' \wedge y' \wedge z' \wedge w_j$ then from (i) this relation implies:
(ii1) $max(x \wedge z', y \wedge z' \wedge w_i) >_\Pi max(x' \wedge z', y' \wedge z' \wedge w_j)$ or
(ii2) $max(x \wedge z', y \wedge z' \wedge w_i) =_\Pi max(x' \wedge z', y' \wedge z' \wedge w_j)$ and
$min(x \wedge z', y \wedge z' \wedge w_i) >_\Pi min(x' \wedge z', y' \wedge z' \wedge w_j)$

Then it is enough to show that $y \wedge z' \wedge w_i =_\Pi y \wedge z'$ and $y' \wedge z' \wedge w_j =_\Pi y' \wedge z'$ in order to show that (i1) and (i2) contradict (ii1) and (ii2).

Let us prove that $y \wedge z' \wedge w_i =_\Pi y \wedge z'$
(the proof of $y' \wedge z' \wedge w_j =_\Pi y' \wedge z'$ is analogous).
By definition we have :
(b) $y \wedge z' =_\Pi max_w\{y \wedge z' \wedge w\} =_\Pi max(y \wedge z' \wedge w_i, max_{w'_i \neq_\Pi w_i}\{y \wedge z' \wedge w'_i\})$
Moreover, recall that w_i maximizes $x \wedge y \wedge z'$ then $\forall w'_i \in D_W$ s.t. $w'_i \neq_\Pi w_i$:
$x \wedge y \wedge z' \wedge w_i \geq_\Pi x \wedge y \wedge z' \wedge w'_i$. Then, for a given $w'_i \neq_\Pi w_i$

- if $x \wedge y \wedge z' \wedge w_i >_\Pi x \wedge y \wedge z' \wedge w'_i$, then from (i), we can distinguish two cases:
  (a) **either** $max(x \wedge z', y \wedge z' \wedge w_i) >_\Pi max(x \wedge z', y \wedge z' \wedge w'_i)$
  $\Rightarrow max(x \wedge z', y \wedge z' \wedge w_i) >_\Pi x \wedge z'$ and $max(x \wedge z', y \wedge z' \wedge w_i) >_\Pi y \wedge z' \wedge w'_i$
  $\Rightarrow y \wedge z' \wedge w_i >_\Pi x \wedge z'$ (otherwise $x \wedge z' >_\Pi x \wedge z'$ which is impossible)
  $\Rightarrow y \wedge z' \wedge w_i >_\Pi y \wedge z' \wedge w'_i$
  (b) **or** $max(x \wedge z', y \wedge z' \wedge w_i) =_\Pi max(x \wedge z', y \wedge z' \wedge w'_i)$ and
  $min(x \wedge z', y \wedge z' \wedge w_i) >_\Pi min(x \wedge z', y \wedge z' \wedge w'_i)$
  $\Rightarrow min(x \wedge z', y \wedge z' \wedge w'_i) <_\Pi x \wedge z'$ and $min(x \wedge z', y \wedge z' \wedge w'_i) <_\Pi y \wedge z' \wedge w_i$
  $\Rightarrow y \wedge z' \wedge w'_i <_\Pi x \wedge z'$ (otherwise $x \wedge z' <_\Pi x \wedge z'$ which is impossible)
  $\Rightarrow y \wedge z' \wedge w_i >_\Pi y \wedge z' \wedge w'_i$
  (Indeed, $x \wedge z' >_\Pi y \wedge z' \wedge w'_i$ and
  $max(x \wedge z', y \wedge z' \wedge w_i) =_\Pi max(x \wedge z', y \wedge z' \wedge w'_i)$
  $\Rightarrow max(x \wedge z', y \wedge z' \wedge w_i) = x \wedge z'$
  $\Rightarrow x \wedge z' \geq_\Pi y \wedge z' \wedge w_i$
  Hence, $min(x \wedge z', y \wedge z' \wedge w_i) >_\Pi min(x \wedge z', y \wedge z \wedge w'_i)$
  $\Rightarrow y \wedge z' \wedge w_i >_\Pi y \wedge z' \wedge w'_i)$
- if $x \wedge y \wedge z' \wedge w_i =_\Pi x' \wedge y' \wedge z' \wedge w'_i$, then from(i) we deduce that:
  (j1) $max(x \wedge z', y \wedge z' \wedge w_i) =_\Pi max(x \wedge z', y \wedge z' \wedge w'_i)$ and
  (j2) $min(x \wedge z', y \wedge z' \wedge w_i) =_\Pi min(x \wedge z', y \wedge z' \wedge w'_i)$
  Let $a = x \wedge z'$ and $b = y \wedge z' \wedge w_i$ and let us show that $(j1) + (j2) \Rightarrow b = c$:
  (a) if $a > b$
    i. if $a > c$ then $(j2) \Rightarrow b = c$
    ii. if $a < c$ then $(j1) \Rightarrow a = c$ and $(j2) \Rightarrow b = a$
    which implies that $b = c$
    iii. $a = c$ then $(j2) \Rightarrow b = c$
  (b) if $a < b$
    i. if $a > c$ then $(j1) \Rightarrow b = a$ and $(j2) \Rightarrow a = c$
    which implies that $b = c$
    ii. if $a < c$ then $(j1) \Rightarrow b = c$
    iii. $a = c$ then $(j1) \Rightarrow b = c$

(c) if $a = b$

i. if $a > c$ then $(j2) \Rightarrow b = c$
ii. if $a < c$ then $(j1) \Rightarrow b = c$
iii. $a = c$ then $(j1) \Rightarrow b = c$

This means that $y \wedge z' \wedge w_i =_\Pi y \wedge z' \wedge w_i'$.

Thus, it is clear that $\forall w_i' \neq_\Pi w_i, y \wedge z' \wedge w_i \geq_\Pi y \wedge z' \wedge w_i'$, so from (b) we deduce that $y \wedge z' =_\Pi y \wedge z' \wedge w_i$.

Case 2: $\exists x, x' \in D_X, \exists y, y' \in D_Y$ s.t. (c) $x \wedge y \wedge z' =_\Pi x' \wedge y' \wedge z'$ but
(i1) $max(x \wedge z', y \wedge z') \neq_\Pi max(x' \wedge z', y' \wedge z')$ or
(i2) $min(x \wedge z', y \wedge z') \neq_\Pi min(x' \wedge z', y' \wedge z')$

From (c) we have $x \wedge y \wedge z' \wedge w_i =_\Pi x' \wedge y' \wedge z' \wedge w_j$ where w_i is one of the instances of W which maximizes $x \wedge y \wedge z'$ and w_j is one of the instances of W which maximizes $x' \wedge y' \wedge z'$.
From (i), $x \wedge y \wedge z' \wedge w_i =_\Pi x' \wedge y' \wedge z' \wedge w_j$ implies:
(ii1) $max(x \wedge z', y \wedge z' \wedge w_i) =_\Pi max(x' \wedge z', y' \wedge z' \wedge w_j)$ and
(ii2) $min(x \wedge z', y \wedge z' \wedge w_i) =_\Pi min(x' \wedge z', y' \wedge z' \wedge w_j)$.
Moreover, we have shown above that $y \wedge z' \wedge w_i =_\Pi y \wedge z'$ and that $y' \wedge z' \wedge w_j =_\Pi y' \wedge z'$ then (i1) and (i2) contradict (ii1) and (ii2).

- Decomposition property for $I_{leximin}$.
We want to prove that
$I_{leximin}(X, Y \mid Z \cup W) \Rightarrow I_{leximin}(X, Y \mid Z)$ and $I_{leximin}(X, W \mid Z)$.
We only prove that (i) $I_{leximin}(X, Y \mid Z \cup W)$ is true, then $I_{leximin}(X, Y \mid Z)$ is true (the proof of (i) $I_{leximin}(X, Y \mid Z \cup W) \Rightarrow I_{leximin}(X, W \mid Z)$ is analogous).
Suppose that $I_{leximin}(X, Y \cup W \mid Z)$ is true but not $I_{leximin}(X, Y \mid Z)$.
Let us consider the two cases where $I_{leximin}(X, Y \mid Z)$ is falsified:

Case 1: $\exists x, x' \in D_X, \exists y, y' \in D_Y, \exists z' \in D_Z$ s.t.
(a) $x \wedge y \wedge z' >_\Pi x' \wedge y' \wedge z'$ but
(i1) $min(x \wedge z', y \wedge z') <_\Pi min(x' \wedge z', y' \wedge z')$ or
(i2) $min(x \wedge z', y \wedge z') =_\Pi min(x' \wedge z', y' \wedge z')$ and
$max(x \wedge z', y \wedge z') \leq_\Pi max(x' \wedge z', y' \wedge z')$

By definition we have $x \wedge y \wedge z' =_\Pi max_w\{x \wedge y \wedge z' \wedge w\}$ and
$x' \wedge y' \wedge z' =_\Pi max_w\{x' \wedge y' \wedge z' \wedge w\}$
Let w_i be one of the instances of W which maximizes $x \wedge y \wedge z'$ and w_j be one of the instances of W which maximizes $x' \wedge y' \wedge z'$, namely:
$x \wedge y \wedge z' =_\Pi x \wedge y \wedge z' \wedge w_i$ and $x' \wedge y' \wedge z' =_\Pi x' \wedge y' \wedge z' \wedge w_j$ From (a) we have

$x \wedge y \wedge z' \wedge w_i >_\Pi x' \wedge y' \wedge z' \wedge w_j$ then from (i) this relation implies:
(ii1) $max(x \wedge z', y \wedge z' \wedge w_i) >_\Pi max(x' \wedge z', y' \wedge z' \wedge w_j)$ or
(ii2) $max(x \wedge z', y \wedge z' \wedge w_i) =_\Pi max(x' \wedge z', y' \wedge z' \wedge w_j)$ and
$min(x \wedge z', y \wedge z' \wedge w_i) >_\Pi min(x' \wedge z', y' \wedge z' \wedge w_j)$

Then it is enough to show that $y \wedge z' \wedge w_i =_\Pi y \wedge z' \wedge z$ and $y' \wedge z' \wedge w_j =_\Pi y' \wedge z'$ in order to prove that (i1) and (i2) contradict (ii1) and (ii2).

Let us prove that $y \wedge z' \wedge w_i =_\Pi y \wedge z'$
(the proof of $y' \wedge z' \wedge w_j =_\Pi y' \wedge z'$ is analogous).
By definition we have :
(b) $y \wedge z' =_\Pi max_w\{y \wedge z' \wedge w\} =_\Pi max(y \wedge z' \wedge w_i, max_{w'_i \neq_\Pi w_i}\{y \wedge z' \wedge w'_i\})$
Moreover w_i maximizes $x \wedge y \wedge z'$ then $\forall w'_i \in D_W$ s.t. $w'_i \neq_\Pi w_i$:
$x \wedge y \wedge z' \wedge w_i \geq_\Pi x \wedge y \wedge z' \wedge w'_i$. Then,

- if $x \wedge y \wedge z' \wedge w_i >_\Pi x \wedge y \wedge z' \wedge w'_i$, then from (i) we can distinguish two cases:
  (a) **either** $min(x \wedge z', y \wedge z' \wedge w_i) >_\Pi min(x \wedge z', y \wedge z' \wedge w'_i)$
  $\Rightarrow min(x \wedge z', y \wedge z' \wedge w'_i) <_\Pi x \wedge z'$ and $min(x \wedge z', y \wedge z' \wedge w'_i) <_\Pi y \wedge z' \wedge w'_i$
  $\Rightarrow y \wedge z' \wedge w'_i <_\Pi x \wedge z'$ (otherwise $x \wedge z' <_\Pi x \wedge z'$)
  $\Rightarrow y \wedge z' \wedge w_i >_\Pi y \wedge z' \wedge w'_i$
  (b) **or** $min(x \wedge z', y \wedge z' \wedge w_i) =_\Pi min(x \wedge z', y \wedge z' \wedge w'_i)$ and
  $max(x \wedge z', y \wedge z' \wedge w_i) >_\Pi max(x \wedge z', y \wedge z' \wedge w'_i)$
  $\Rightarrow max(x \wedge z', y \wedge z' \wedge w_i) >_\Pi x \wedge z'$ and $max(x \wedge z', y \wedge z' \wedge w_i) >_\Pi y \wedge z' \wedge w'_i$
  $\Rightarrow y \wedge z' \wedge w_i >_\Pi x \wedge z'$ (otherwise $x \wedge z' >_\Pi x \wedge z'$)
  $\Rightarrow y \wedge z' \wedge w_i >_\Pi y \wedge z' \wedge w'_i$
- if $x \wedge y \wedge z' \wedge w_i =_\Pi x' \wedge y' \wedge z' \wedge w'_i$, then from (i) we deduce that:
  $min(x \wedge z', y \wedge z' \wedge w_i) =_\Pi min(x \wedge z', y \wedge z' \wedge w'_i)$ and
  $max(x \wedge z', y \wedge z' \wedge w_i) =_\Pi max(x \wedge z', y \wedge z' \wedge w'_i)$
  $\Rightarrow y \wedge z' \wedge w_i =_\Pi y \wedge z' \wedge w'_i$ (in the same manner than in the previous proof)

Thus, it is clear that $\forall w'_i \neq_\Pi w_i, y \wedge z' \wedge w_i \geq_\Pi y \wedge z' \wedge w'_i$, so from (b) we deduce that $y \wedge z' =_\Pi y \wedge z' \wedge w_i$.

Case 2: $\exists x, x' \in D_X, \exists y, y' \in D_Y, \exists z' \in D_Z$ s.t.
(b) $x \wedge y \wedge z' =_\Pi x' \wedge y' \wedge z'$ but
(i1) $min(x \wedge z', y \wedge z') \neq_\Pi min(x' \wedge z', y' \wedge z')$ or
(i2) $max(x \wedge z', y \wedge z') \neq_\Pi max(x' \wedge z', y' \wedge z')$

From (b) we have $x \wedge y \wedge z' \wedge w_i =_\Pi x' \wedge y' \wedge z' \wedge w_j$ where w_i is one of the instances of W which maximizes $x \wedge y \wedge z'$ and w_j is one of the instances of W which maximizes $x' \wedge y' \wedge z'$.
From (i) $x \wedge y \wedge z' \wedge w_i =_\Pi x' \wedge y' \wedge z' \wedge w_j$ implies:
(ii1) $min(x \wedge z', y \wedge z' \wedge w_i) =_\Pi min(x' \wedge z', y' \wedge z' \wedge w_j)$ and
(ii2) $max(x \wedge z', y \wedge z' \wedge w_i) =_\Pi max(x' \wedge z', y' \wedge z' \wedge w_j)$.
Moreover, we have shown above that $y \wedge z' \wedge w_i =_\Pi y \wedge z'$ and that $y' \wedge z' \wedge w_j =_\Pi y' \wedge z'$ then (i1) and (i2) contradict (ii1) and (ii2).

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